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20

USAAVLABS TECHNICAL REPORT 69-64

ELASTIC AND MAXIMUM STRENGTH ANALYSES OF POSTBUCKLED RECTANGULAR PLATES BASED UPON MODIFIED VERSIONS OF REISSNER'S VARIATIONAL PRINCIPLE

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July 1970

U. S. ARMY AVIATION MATERIEL LABORATORIES
FORT EUSTIS, VIRGINIA

CONTRACT DA 44-177-AMC-115(T)
DEPARTMENT OF AERONAUTICS AND ASTRONAUTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

76

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2

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This research was carried out under Contract DA 44-177-AMC-115(T) with Stanford University.

The data presented in this report are a result of research conducted to investigate postbuckling behavior and maximum strength of uniformly shortened, simply supported rectangular plates with straight, unloaded edges by use of Reissner's variational principle.

This report has been reviewed by the U. S. Army Aviation Materiel Laboratories and is considered to be technically sound. It is published for the exchange of information and the stimulation of future research.

Task 1F162204A17002
Contract DA 44-177-AMC-115(T)
USAAVLABS Technical Report 69-64
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SUMMARY

The postbuckling and maximum strength analyses of uniformly shortened, simply supported rectangular plates with straight, unloaded edges are performed by using Reissner's variational principle in conjunction with a deformation theory of plasticity. The results are compared to (1) experimental data for a rectangular plate in which the test conditions basically reflect the boundary conditions specified in the present analyses, and (2) a potential energy solution that correlates well with experiment, but in which the effect of waveform change on the average stress carried by the plate is accounted for only in a gross manner and in which the effect of small, local unloading is neglected. Good agreement with experiment establishes confidence in the new approach and indicates that the simplified technique utilized in the potential energy solution to compensate for waveform changes may be employed for engineering purposes. Both analytical approaches predict a slightly conservative plate maximum strength relative to the experimental result. This discrepancy is attributed to slight departures of the test conditions from the ideal boundary and loading conditions assumed in the analyses.

An application of Reissner's principle and a modified version of the principle is undertaken initially to obtain the elastic postbuckling behavior of uniaxially compressed square and rectangular plates. Excellent agreement of elastic-solution results with essentially "exact" solutions of other authors for the same boundary conditions establishes the effectiveness of the Reissner and modified Reissner principles and justifies the application of the Reissner-principle approach to the maximum strength problem.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

LIST OF TABLES

LIST OF SYMBOLS

A_i, B_i, C	in-plane stress parameters
A_{ij}, B_{ij}, C_{ij}	bending stress parameters
a_{ij}, b_{ij}, c_{ij}	in-plane stress parameters
b	plate dimension in y-direction, in.
D	flexural rigidity of solid plate = $\dfrac{Et^3}{12(1-\mu^2)}$, lb-in.
D_s	flexural rigidity of two-element plate = $\dfrac{2Eh^2t_f}{3}$, $(\mu = \frac{1}{2})$, lb-in.
E	Young's modulus for plate material, psi
E_s	secant modulus, psi
e	unit shortening applied to plate, in./in.
e_{cr}	buckling strain or shortening, in./in.
F'	complementary energy density = $\int^{\sigma_{eff}} \epsilon_{eff} d\sigma_{eff}$, psi
f	displacement parameter
h	overall thickness of two-element plate measured between center lines of faces, in.
i, j, m, n	integers
K	material constant appearing in Equation (7)
$K_1 \ldots K_{45}$	numerical constants
L	plate dimension in x-direction, in.
M_x, M_y	bending moment per unit length, lb
M_{xy}	twisting moment per unit length, lb
m	number of buckles in x-direction
n	material constant appearing in Equation (7)
t	thickness of solid plate, in.
t_f	thickness of one face of two-element plate, in.

U''	Reissner functional, lb-in.
u,v,w	displacement of point in middle surface of plate in x-, y-, and z-directions, respectively, in.
$\overline{W}_{ij}$	nondimensionalized displacement parameters
V	volume of plate, in.^{3}
x,y,z	plate coordinates (see Figure 1), in.
α	bending stress parameter = $A_{13}/A_{11} = B_{13}/B_{11} = C_{13}/C_{11}$
β	buckle aspect ratio = b/λ
γ_{xy}	total shear strain in xy-plane, in./in.
γ'_{xy}	shearing strain at middle surface, in./in.
γ''_{xy}	shearing strain due to twisting, in./in.
ϵ_{eff}	effective strain = $\frac{2}{\sqrt{3}}\sqrt{\epsilon_x^2 + \epsilon_y^2 + \epsilon_x\epsilon_y + \frac{\gamma_{xy}^2}{4}}$, in./in.
ϵ_x, ϵ_y	total components of strain in x- and y-directions, respectively, in./in.
ϵ'_x, ϵ'_y	components of strain at middle surface in x- and y-directions, respectively, in./in.
$\epsilon''_x, \epsilon''_y$	components of strain due to bending in x- and y-directions, respectively, in./in.
η	nondimensional lateral coordinate, $2y/b$
λ	buckle half wavelength in x-direction, in.
μ	Poisson's ratio for plate material
ξ	nondimensional axial coordinate, $2x/\lambda$
σ_{av}	average compressive stress, psi
σ_{cr}	compressive buckling stress, psi
σ_{cy}	compressive yield stress of material, psi
σ_{eff}	effective stress = $\sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2}$, psi

σ_{max}	maximum average compressive stress, psi
σ_x, σ_y	components of stress in x- and y-directions, respectively, psi
σ_x', σ_y'	local average stress in x- and y-directions, respectively, psi
σ_x'', σ_y''	bending stress in x- and y-directions, respectively, psi
τ_{xy}	total shear stress in xy-plane, psi
τ_{xy}'	local average shear stress, psi
τ_{xy}''	shear stress due to twisting, psi

INTRODUCTION

The analytical determination of postbuckled plate behavior in the plastic range has been investigated by Mayers and Budiansky for square plates in Reference 1 and by Mayers, Nelson, and Smith for rectangular plates in Reference 2. The latter study serves to explain the absence of a plate maximum strength in the case of the square-plate analysis of Reference 1 and correlates well with the experimental data presented by Stein in Reference 3. In Reference 4, Stein discusses the phenomenon of buckle wavelength change in elastic structures; in Reference 3, a purely theoretical elastic analysis implies that, unlike square plates, postbuckled rectangular plates can change buckle aspect ratio at reasonable values of end shortening. The experimental evidence presented in Reference 3 proves the phenomenon to be quite true; in addition, the experimental data reflect a maximum load which cannot be predicted by the accompanying elastic analysis.

Although the analysis of Reference 2 correlates well with the plate maximum strength data of Reference 3, the method for obtaining the final load-shortening curves leaves a question unanswered. This question concerns the validity of the approximate technique of accounting for an accurate postbuckled waveform of the plate by adjusting the first-approximation load-shortening curve at a given shortening in the plastic range by the ratio, at the same shortening value, of the "exact" elastic load-shortening curve to the first-approximation elastic load-shortening curve. By first-approximation load-shortening curve is meant that curve obtained under the assumption that the waveform at initial buckling persists through the postbuckling range to plate failure. This technique provides good correlation of the theoretical prediction of Reference 2 for the maximum strength and number of buckles at failure and experimental results of Reference 3. This procedure is highly desirable for engineering purposes, as the complexity of the analysis is reduced considerably; for problems in which an accurate elastic analysis exists, a rapid means is provided, therefore, for evaluating the maximum strength of plate elements. Essentially "exact" elastic solutions to the problem of a compressed, simply supported flat plate with straight, unloaded edges are given by Stein in Reference 3 (square and rectangular plates) and by Levy in Reference 5 (square plates).

One way in which to justify the use of the load-shortening curve adjustment procedure, on other than an experimental correlation basis, is to extend the potential energy solution of Reference 2 by utilizing sufficient terms in the functions describing the displacements u, v, and w such that the essentially exact elastic solution can be obtained and extended into the inelastic region. This procedure would not require that any adjustment be made to the inelastic solution, since the waveform beyond initial buckling would be adequately described.

An alternate method is available, however, that reduces the complexity of the problem considerably. This method is based on the variational principle of Reissner given in Reference 6. Stated briefly, the principle is based on the simultaneous vanishing of the first variation of the Reissner functional with respect to admissible states of stress and strain. The variation yields Euler equations corresponding to (1) stress-displacement relations and (2) equilibrium conditions, and the associated boundary conditions. Appendix I contains the variation of the Reissner functional for a two-element plate with a core, rigid in shear, separating the faces. This configuration, employed in References 1 and 2 to eliminate the added complexity of plasticity effects through the plate thickness, is utilized again herein for comparison purposes.

Three basic advantages of employing Reissner's principle rather than the more conventional minimum potential energy theorem for the solution of the inelastic plate problem are:

1. In the former, stresses and displacements are assumed as completely independent quantities in a Rayleigh-Ritz solution, while in the latter, displacements must be determined to great accuracy in order to obtain satisfactory stresses through the particular constitutive relations being used.
2. A first approximation to the inelastic stress distribution can be obtained from an accurate elastic analysis; thus, the

displacement distributions are no longer required to high accuracy. The fact that the stress-displacement relations given by the variation of the Reissner functional are satisfied only approximately, and, perhaps, unsatisfactorily, by a Rayleigh-Ritz solution is of little consequence. This is due to the proper, independent selection of stresses and displacements which, in turn, eliminates the need to evaluate stresses in terms of inaccurate displacement derivatives through the constitutive relations. With equilibrium and compatibility of deformations satisfied in either case (potential energy method or Reissner's principle), it would appear that an unacceptable stress distribution with the constitutive relations satisfied is much less desirable than reasonably correct displacement and stress distributions with the constitutive law not satisfied exactly. This conjecture is strengthened even more when it is considered that the constitutive law assumed to hold even in elastic, homogeneous, isotropic plate theory is only a good approximation for a conventional material. For either non-linear elastic or inelastic considerations, the desirability of seeking exact satisfaction of an idealized constitutive law should be subservient to insuring that displacement compatibility and equilibrium are satisfied.

3. The effects of nonlinear stress-strain relations are readily incorporated into the Reissner functional by introducing the Ramberg-Osgood representation of the material stress-strain curve (Reference 7). With reference to the remarks in item 2, it should be noted that the uniaxial stress-strain curve is an accurate representation in general only up to the region of the 0.2% offset yield stress.

Strangely enough, Reissner's principle has not been widely utilized to date. One application of the theorem is found in Reference 8, in which Sanders, McComb, and Schlechte extend the principle to consider creep behavior with specific applications to plates and columns. In Reference 9, Mayers and Rehfield apply a modified version of the principle to study the postbuckling behavior of axially compressed circular cylindrical shells. The modification of the principle consists of coupling the bending stresses or their resulting moments to the curvature through Hooke's law, thus leaving only the in-plane stresses, in-plane displacements, and lateral displacements as variationally independent quantities.

The present analysis also employs the modified principle for the elastic postbuckled plate problem, while comparing solutions based upon two- and three-term lateral displacement functions to the essentially "exact" square-plate solutions of Levy and Stein. Both the modified and the unmodified Reissner functionals are employed with solutions compared to those of Stein for rectangular plates with aspect ratios 1.33, 1.5, and 2.0 . In all cases, there is little difference obtained in the quantitative results representing the load-shortening relationships.

With confidence established in the present method of solution, the inelastic postbuckling solution is obtained for a rectangular plate of aspect ratio 5:1 . Comparison is made to the solution of Reference 2 and the experimental data of Reference 3. Good agreement is obtained, with the results indicating that the load-shortening curve adjustment procedure employed in Reference 2 is fully justified for predicting plate maximum strength in the respective analyses of References 1 and 2.

THEORY

STATEMENT OF PROBLEM AND BASIC ASSUMPTIONS

The problem considered herein is that of the elastic and inelastic post-buckling behavior of uniformly shortened, simply supported square and rectangular plates that buckle initially in the elastic range and that have straight, unloaded edges free to translate in the plane of the plate. As observed in Reference 2, major factors in approximating satisfactorily the postbuckling behavior and, ultimately, the maximum strength of uniformly compressed, simply supported rectangular plates with straight, unloaded edges are the combined effects of (1) changes in waveform beyond initial buckling, (2) changes in buckle aspect ratio beyond initial buckling, and (3) plasticity. The emphasis is placed on determining the maximum strength of rectangular plates without utilizing the load-shortening curve adjustment procedure of References 1 and 2. The adjustment procedure, in gross fashion, accounts for change in buckle waveform as postbuckling proceeds. A new solution of the maximum strength problem is obtained by employing Reissner's variational theorem; the results of the present investigation and that of Reference 2 are compared, and the accuracy of the load-shortening curve adjustment procedure is evaluated. To establish confidence in the inelastic analysis based upon the Reissner principle, elastic solutions for the postbuckling problem are undertaken and compared with essentially "exact" solutions presented in References 3 and 5.

The inclusion of plasticity effects in conjunction with analysis based upon the small strain, moderate rotations strain-displacement relations of the von Kármán plate theory results in a problem that is nonlinear in two respects. The two-element plate description (see Figure 1), introduced in References 1 and 2 to avoid complexity associated with accounting for plasticity effects through the plate thickness, is employed herein to provide a direct comparison of the results of the present analyses with those of Reference 2. The core separating the faces of the plate is considered to be rigid in shear in the present investigation, but it provides a means to extend the current analyses to include transverse shear deformations when the postbuckling behavior of sandwich plates is of interest. Again, plastic behavior is based upon deformation theory, and the ramifications of local unloading are assessed in retrospect. Material compressibility is neglected; thus, Poisson's ratio is taken as 1/2 throughout the present development.

VARIATIONAL PRINCIPLE

In essence, Reissner's principle states that the vanishing of the first variation of the Reissner functional with respect to admissible states of stress and strain establishes the stress-displacement relations, the equilibrium equations and the associated boundary conditions.

The general functional is formulated first for the inelastic problem; specializations are made subsequently for purely elastic considerations in both an unmodified and a modified version of the functional. The functional modification corresponds to the one introduced in Reference 9 to couple the bending stresses to the assumed deflections and to leave only the membrane stresses to be determined from the variational process.

The functional for plate-type considerations with edge displacements specified is given as

$$U'' = \int_V \left\{ \sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau_{xy} \gamma_{xy} - F' \right\} dV \qquad (1)$$

where F' is the complementary energy density. If the end loading were to be prescribed rather than the end shortening, then the Reissner functional would include another functional term corresponding to the potential of the prescribed edge loads, as in the potential energy principle. The vanishing of the first variation of the functional expressed in Equation (1) is performed in Appendix I for the two-element plate considered herein. The variation establishes Euler equations corresponding to (1) in-plane and out-of-plane equilibrium given in Equations (35) and (36), respectively, and (2) stress-displacement and moment-curvature relations given in Equations (34a) and (34b), respectively. Stress boundary conditions for the uniformly compressed, simply supported plate with straight, unloaded edges resulting from the variation are given in Equation (37).

It is convenient to obtain the functional of Equation (1) in terms of stresses and strains such that the nonlinear portion of the functional due to plasticity is separable from the elastic portion. To this end, the complementary energy density F' is first written incrementally as

$$dF' = \epsilon_x d\sigma_x + \epsilon_y d\sigma_y + \gamma_{xy} d\tau_{xy} \tag{2}$$

Next, the effective stress and strain quantities

$$\epsilon_{eff} = \frac{2}{\sqrt{3}} \sqrt{\epsilon_x^2 + \epsilon_y^2 + \epsilon_x \epsilon_y + \frac{\gamma_{xy}^2}{4}} \tag{3}$$

and

$$\sigma_{eff} = \sqrt{\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2} \tag{4}$$

which are related by

$$\frac{\sigma_{eff}}{\epsilon_{eff}} = E_s \tag{5}$$

are introduced. It can be shown readily that

$$\epsilon_{eff} d\sigma_{eff} = dF'$$

by utilizing the secant modulus theory stress-strain relations for an incompressible material of either Reference 1 or Reference 2. The Reissner functional may be written then as

$$U'' = \int_V \left\{ \sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau_{xy} \gamma_{xy} - \int_0^{\sigma_{eff}} \epsilon_{eff} d\sigma_{eff} \right\} dV \tag{6}$$

Equation (6) can be simplified further by employing the Ramberg-Osgood three-parameter representation of the uniaxial stress-strain curve originally developed for aluminum alloy, stainless steel, and carbon-sheet steel in Reference 7 and given by

$$\epsilon_{eff} = \frac{\sigma_{eff}}{E} + K\left(\frac{\sigma_{eff}}{E}\right)^n \tag{7}$$

where E , K , and n are constants determined for each material under consideration. The stress-strain curve (see Figure 2) for aluminum 2024-T3 used in the present inelastic analysis is described by $K = 3.14 \times 10^{17}$, $n = 8.60$, and $E = 10.7 \times 10^6$ psi . Substitution of Equation (7) into Equation (6) and subsequent integration yields

$$U'' = \int_V \left\{ \sigma_x\epsilon_x + \sigma_y\epsilon_y + \tau_{xy}\gamma_{xy} - \left[\frac{\sigma_{eff}^2}{2E} + \frac{KE}{n+1}\left(\frac{\sigma_{eff}}{E}\right)^{n+1}\right]\right\} dV \tag{8}$$

Upon integration over the plate thickness, the volume integral of Equation (8) for the two-element plate becomes, with $V = 2t_f Lb$,

$$\frac{U''}{EV} = \int_0^1\int_0^1 \left\{\left(\frac{\sigma_x'}{E}\right)\epsilon_x' + \left(\frac{\sigma_y'}{E}\right)\epsilon_y' + \left(\frac{\tau_{xy}'}{E}\right)\gamma_{xy}' + \left(\frac{\sigma_x''}{E}\right)\epsilon_x'' + \left(\frac{\sigma_y''}{E}\right)\epsilon_y''\right.$$

$$+ \left(\frac{\tau_{xy}''}{E}\right)\gamma_{xy}'' - \frac{1}{2}\left[\frac{1}{2}\left(\frac{\sigma_{eff}}{E}\right)^2 + \frac{K}{n+1}\left(\frac{\sigma_{eff}}{E}\right)^{n+1}\right]_t$$

$$\left. - \frac{1}{2}\left[\frac{1}{2}\left(\frac{\sigma_{eff}}{E}\right)^2 + \frac{K}{n+1}\left(\frac{\sigma_{eff}}{E}\right)^{n+1}\right]_b \right\} d\xi d\eta \tag{9}$$

where the nondimensional coordinates $\xi = 2x/\lambda$ and $\eta = 2y/b$ are introduced, and the total stresses and strains are written in terms of in-plane and bending contributions as denoted by the primed and double-primed quantities, respectively. The subscript t refers to the top face, and the subscript b refers to the bottom face. Finally, through the utilization of the strain-displacement relations of von Kármán plate theory, modified for the two-element plate as indicated in Appendix I, Equation (27), the Reissner functional given in Equation (9) becomes

$$\frac{U''}{EV} = \int_0^1 \int_0^1 \Bigg\{ \left(\frac{\sigma_x'}{E}\right)\left[u_{,x} + \frac{1}{2} w_{,x}^2\right] + \left(\frac{\sigma_y'}{E}\right)\left[v_{,y} + \frac{1}{2} w_{,y}^2\right]$$

$$+ \left(\frac{\tau_{xy}'}{E}\right)\left[v_{,x} + u_{,y} + w_{,x}w_{,y}\right] + \left(\frac{\sigma_x''}{E}\right)\left[\frac{h}{2} w_{,xx}\right] + \left(\frac{\sigma_y''}{E}\right)\left[\frac{h}{2} w_{,yy}\right]$$

$$+ \left(\frac{\tau_{xy}''}{E}\right)\left[h\, w_{,xy}\right] - \frac{1}{4}\left[\left(\frac{\sigma_{eff}}{E}\right)_t^2 + \left(\frac{\sigma_{eff}}{E}\right)_b^2\right]$$

$$- \frac{K}{2(n+1)}\left[\left(\frac{\sigma_{eff}}{E}\right)_t^{n+1} + \left(\frac{\sigma_{eff}}{E}\right)_b^{n+1}\right]\Bigg\}\, d\xi d\eta \qquad (10)$$

Specialization for Elastic Behavior, Unmodified Version

The Reissner functional given in Equation (1) is specialized for the elastic postbuckling problems simply by setting K equal to 0 . The effective stress, in general,

$$\sigma_{eff}^2 = \sigma_x^2 + \sigma_y^2 - \sigma_x\sigma_y + 3\tau_{xy}^2 \qquad (11)$$

may be written for the two-element plate as

$$(\sigma_{eff}^2)_{t,b} = (\sigma_x' \pm \sigma_x'')^2 + (\sigma_y' \pm \sigma_y'')^2 - (\sigma_x' \pm \sigma_x'')(\sigma_y' \pm \sigma_y'')$$

$$+ 3\,(\tau_{xy}' \pm \tau_{xy}'')^2 \qquad (12)$$

It is easily verified that

$$\frac{1}{4}\left[\left(\frac{\sigma_{eff}}{E}\right)_t^2 + \left(\frac{\sigma_{eff}}{E}\right)_b^2\right] = \frac{1}{2}\left[\left(\frac{\sigma_x'}{E}\right)^2 + \left(\frac{\sigma_y'}{E}\right)^2 - \left(\frac{\sigma_x'}{E}\right)\left(\frac{\sigma_y'}{E}\right) + 3\left(\frac{\tau_{xy}'}{E}\right)^2\right.$$

$$\left. + \left(\frac{\sigma_x''}{E}\right)^2 + \left(\frac{\sigma_y''}{E}\right)^2 - \left(\frac{\sigma_x''}{E}\right)\left(\frac{\sigma_y''}{E}\right) + 3\left(\frac{\tau_{xy}''}{E}\right)^2\right] \qquad (13)$$

Thus, Equation (10), with K equal to 0, can be written

$$\frac{U''}{EV} = \int_0^1\int_0^1 \left\{\left(\frac{\sigma_x'}{E}\right)\left[u_{,x} + \frac{1}{2}w_{,x}^2\right] + \left(\frac{\sigma_y'}{E}\right)\left[v_{,y} + \frac{1}{2}w_{,y}^2\right]\right.$$

$$+ \left(\frac{\tau_{xy}'}{E}\right)\left[v_{,x} + u_{,y} + w_{,x}w_{,y}\right] + \left(\frac{\sigma_x''}{E}\right)\left[\frac{h}{2}w_{,xx}\right] + \left(\frac{\sigma_y''}{E}\right)\left[\frac{h}{2}w_{,yy}\right]$$

$$+ \left(\frac{\tau_{xy}''}{E}\right)\left[hw_{,xy}\right] - \frac{1}{2}\left[\left(\frac{\sigma_x'}{E}\right)^2 + \left(\frac{\sigma_y'}{E}\right)^2 - \left(\frac{\sigma_x'}{E}\right)\left(\frac{\sigma_y'}{E}\right) + 3\left(\frac{\tau_{xy}'}{E}\right)^2\right.$$

$$\left.\left. + \left(\frac{\sigma_x''}{E}\right)^2 + \left(\frac{\sigma_y''}{E}\right)^2 - \left(\frac{\sigma_x''}{E}\right)\left(\frac{\sigma_y''}{E}\right) + 3\left(\frac{\tau_{xy}''}{E}\right)^2\right]\right\} d\xi d\eta \qquad (14)$$

The formulation of the Reissner functional given in Equation (14) permits an independent selection of the stress and displacement quantities that satisfy at least the prescribed displacement boundary conditions implied in Appendix I for application of the Rayleigh-Ritz procedure. However, since the stress distributions are selected independently in the Reissner formulation, the stress boundary conditions may be satisfied as well. The requirement that the functional be stationary with respect to the free parameters which describe the assumed states of stress and displacement then approximately satisfies the equilibrium conditions [Equations

(35) and (36)] and the stress-displacement and moment-curvature relations [Equation (34)].

Specialization for Elastic Behavior, Modified Version

The formulation given in Equation (14) is modified by enforcing, at the outset, the moment-curvature relations given in Equation (34b), which results in the expressions

$$\left.\begin{aligned} \frac{\sigma_x''}{E} &= \frac{2h}{3}\left[w_{,xx} + \frac{1}{2}w_{,yy}\right] \\ \frac{\sigma_y''}{E} &= \frac{2h}{3}\left[w_{,yy} + \frac{1}{2}w_{,xx}\right] \\ \frac{\tau_{xy}''}{E} &= \frac{h}{2}\left[w_{,xy}\right] \end{aligned}\right\} \tag{15}$$

Utilization of the relations given in Equation (15) permits the out-of-plane equilibrium Equation (36) to be written

$$D_s\nabla^4 w - 2t_f(\sigma_x' w_{,xx} + \sigma_y' w_{,yy} + 2\tau_{xy}' w_{,xy}) = 0 \tag{16}$$

Equation (16) is analogous to the familiar fourth-order out-of-plane equilibrium equation for solid plates under the action of in-plane forces given by

$$D\nabla^4 w - t(\sigma_x w_{,xx} + \sigma_y w_{,yy} + 2\tau_{xy} w_{,xy}) = 0 \tag{17}$$

Substitution of the expressions given by Equation (15) into Equation (14) yields the modified version of the functional

$$\frac{U''}{EV} = \int_0^1\!\!\int_0^1 \left\{ \left(\frac{\sigma_x'}{E}\right)\left[u_{,x} + \frac{1}{2} w_{,x}^2\right] + \left(\frac{\sigma_y'}{E}\right)\left[v_{,y} + \frac{1}{2} w_{,y}^2\right] \right.$$

$$+ \left(\frac{\tau_{xy}'}{E}\right)\left[v_{,x} + u_{,y} + w_{,x}w_{,y}\right] - \frac{1}{2}\left[\left(\frac{\sigma_x'}{E}\right)^2 + \left(\frac{\sigma_y'}{E}\right)^2 - \left(\frac{\sigma_x'}{E}\right)\left(\frac{\sigma_y'}{E}\right)\right.$$

$$\left. + 3\left(\frac{\tau_{xy}'}{E}\right)^2\right] + \frac{h^2}{6}\left[\left[w_{,xx}^2 + w_{,yy}^2 + w_{,xx}w_{,yy} + w_{,xy}^2\right]\right] \Bigg\} d\xi d\eta \qquad (18)$$

in which only the quantities σ_x' , σ_y' , τ_{xy}' , u , v , and w are now variationally independent.

METHOD OF SOLUTION

The most efficient application of Reissner's variational principle to obtain solutions of the inelastic postbuckled plate problem utilizing a direct (Rayleigh-Ritz) approach requires prior knowledge of the elastic stress and displacement distributions. The fact that the plates considered herein buckle elastically suggests, with a high degree of confidence, the use of stress and displacement distributions from existing essentially "exact" elastic plate solutions as guides in selecting the assumed distributions for postbuckling into the inelastic range.

In general, based on the quite accurate elastic stress distributions depicted in Reference 5, the in-plane stress system is seen to be suitably described by expressions of the form

$$\left.\begin{aligned} \frac{\sigma_x'}{E} &= \sum_{\text{even}}\sum a_{ij}x^i y^j \\ \frac{\sigma_y'}{E} &= \sum_{\text{even}}\sum b_{ij}x^i y^j \\ \frac{\tau_{xy}'}{E} &= \sum_{\text{odd}}\sum c_{ij}x^i y^j \end{aligned}\right\} \qquad (19)$$

For the functional given in Equation (14), the bending stresses σ_x'', σ_y'', and τ_{xy}'' are, of course, expressible in a similar manner. However, it is found that best results for both accuracy and simplicity are obtained in the present problem by selecting the bending stresses in the <u>form</u> of curvatures, as stipulated by Equation (15), but with perfectly free amplitude coefficients. The modified functional given in Equation (18) for elastic behavior requires no specification of the bending stresses, since the moment-curvature relations are enforced at the outset through Hooke's law.

The displacements satisfying all geometric boundary conditions (see Appendix I) are suitably expressed by series of the form

$$\frac{u}{b} = -e\left(\frac{x}{b}\right) + \sum_{\text{even}}\sum u_{mn} \sin\frac{m\pi x}{\lambda} \cos\frac{n\pi y}{b} \tag{20a}$$

$$\frac{v}{b} = f\left(\frac{y}{b}\right) + \sum_{\text{even}}\sum v_{mn} \cos\frac{m\pi x}{\lambda} \sin\frac{n\pi y}{b} \tag{20b}$$

$$\frac{w}{b} = \sum_{\text{odd}}\sum w_{mn} \cos\frac{m\pi x}{\lambda} \cos\frac{n\pi y}{b} \tag{20c}$$

where e is the applied unit shortening.

The coefficients f, u_{mn}, v_{mn}, w_{mn}, a_{ij}, b_{ij}, and c_{ij} are determined from the condition that $\delta U'' = 0$ with respect to these coefficients. The number of free variables, indicated by Equations (19) and (20), to suitably describe the stresses and displacements may seem excessive at first glance; however, as observed in Reference 9, the in-plane stress distribution can be selected such that one or both of the in-plane equilibrium equations are satisfied independently of the magnitude of the free stress coefficients. If the y-direction in-plane equilibrium equation (35b) is satisfied independently of the stress-coefficient magnitudes, then the v-displacement cannot be specified arbitrarily. This is a consequence of the fact that the satisfaction of the y-direction in-plane equilibrium equation, which is performed at the outset, does not permit the variation $\delta_v U'' = 0$ to be made. Similarly, the u-displacement cannot be specified in terms of unknown amplitudes if the x-direction in-plane equilibrium Equation (35a) is satisfied independently of the stress-coefficient magnitudes. The procedure, wherein one or both of the in-plane equilibriums are satisfied independently of the stress coefficients, has merit in problems where a stress-function approach is not feasible. Such is the case, for example, in Reference 9, where a highly nonlinear elastic shell theory is employed, and in the present problem, where inelastic behavior is present.

ELASTIC PROBLEM, MODIFIED REISSNER FUNCTIONAL (SOLUTIONS A)

The in-plane stress distribution of the form given in Equation (19) is assumed initially, for the present analysis, as

$$\left.\begin{aligned}
\frac{\sigma_x'}{E} &= a_{00} + a_{02}\eta^2 + a_{04}\eta^4 + a_{22}\xi^2\eta^2 + a_{24}\xi^2\eta^4 + a_{42}\xi^4\eta^2 + a_{44}\xi^4\eta^4 \\
\frac{\sigma_y'}{E} &= b_{00} + b_{20}\xi^2 + b_{40}\xi^4 + b_{04}\eta^4 + b_{06}\eta^6 + b_{24}\xi^2\eta^4 + b_{26}\xi^2\eta^6 \\
\frac{\tau_{xy}'}{E} &= c_{13}\xi\eta^3 + c_{15}\xi\eta^5 + c_{33}\xi^3\eta^3 + c_{35}\xi^3\eta^5
\end{aligned}\right\} \quad (21)$$

Due to the large number of stress coefficients, it is convenient at this point to invoke the stress conditions at the plate edges implicit in Equation (33) and thereby reduce the number of free variables considerably. The satisfaction of the stress boundary conditions is not actually necessary, since the surface stresses are not specified for the present problem. This procedure eliminates 8 stress coefficients and reduces the number of free coefficients to 10 . In view of the discussion in the previous section, a further reduction in the number of free stress coefficients can be made by taking the assumed u- and v-displacement functions to be independent of free parameters; thus, it is possible for the in-plane equilibrium Equation (35) to be satisfied identically. With the end shortening specified, a u-displacement function which satisfies the geometric boundary conditions and which contains no free parameters is

$$\frac{u}{b} = -e\left(\frac{x}{b}\right)$$

Immediately, Equation (35a) can be satisfied identically by relating the coefficients of σ_x' and τ_{xy}' .

Now, if the constant v-displacement at the unloaded edges were to be specified with regard to magnitude, then the function

$$\frac{v}{b} = f\left(\frac{y}{b}\right)$$

would satisfy the geometric boundary conditions at $y = \pm b/2$ and would permit the coefficients of σ_y' and τ_{xy}' to be related such that Equation (35b) could be satisfied identically. However, although the v-displacement is constant on the unloaded edges, its magnitude f is actually unspecified for the present problem. Indeed, the magnitude of f should be established in the variational process, with Equation (35b) being satisfied only approximately. Nevertheless, in view of the small influence of f reflected in the solutions presented in References 1 and 2, it is found that the specification of $f = 0$ and the identical satisfaction of Equation (35b), in addition to the identical satisfaction of Equation (35a), lead to accurate results. Thus, Equation (21) can be reduced to the expressions containing only six free variables given by

$$\left.\begin{aligned}
\frac{\sigma_x'}{E} &= A_1 + A_2\eta^2 + A_3\eta^4 + C\left(\xi^2 - \frac{\xi^4}{2}\right)\left(\eta^2 - \frac{5}{3}\eta^4\right) \\
\frac{\sigma_y'}{E} &= B_2\left(\xi^2 - \frac{1}{3}\right) + B_3\left(\xi^4 - \frac{1}{5}\right) - C\beta^2\left(\xi^2 - \frac{1}{3}\right)\left(\frac{\eta^4}{2} - \frac{\eta^6}{3}\right) \\
\frac{\tau_{xy}'}{E} &= -\frac{2}{3}\beta C(\xi - \xi^3)(\eta^3 - \eta^5)
\end{aligned}\right\} \quad (22)$$

where new symbols for the stress coefficients are introduced for simplicity.

There are no restrictions on the out-of-plane displacement w other than those associated with the geometric boundary conditions. As a result, the assumed function for w that satisfies the geometric boundary conditions and that provides sufficient freedom to change the waveform as postbuckling progresses is obtained by retaining only the w_{11}, w_{31}, and w_{13} terms of the out-of-plane displacement series given in Equation (20c). The actual displacement distributions, with both in-plane equilibrium and stress boundary conditions satisfied, are assumed throughout the present analyses to be

$$\left.\begin{aligned}
\frac{u}{b} &= -e\left(\frac{x}{b}\right)\\
\frac{v}{b} &= 0\\
\frac{w}{b} &= w_{11}\cos\frac{\pi x}{\lambda}\cos\frac{\pi y}{b} + w_{31}\cos\frac{3\pi x}{\lambda}\cos\frac{\pi y}{b} + w_{13}\cos\frac{\pi x}{\lambda}\cos\frac{3\pi y}{b}
\end{aligned}\right\}\quad(23)$$

The modified Reissner functional given in Equation (18), employing the expressions given in Equations (22) and (23), is expanded to yield Equation (39) of Appendix II. The solution obtained by utilization of the functional given in Equation (39) with $(\pi h/b)\, w_{13} = \bar{w}_{13} = 0$ is designated as Solution A1, while the solutions obtained with $\bar{w}_{13}$ retained are designated as Solutions A2 .

ELASTIC PROBLEM, UNMODIFIED REISSNER FUNCTIONAL (SOLUTIONS B)

Instead of relating the bending stresses to the curvatures through Hooke's law, as is the case in Solutions A, the bending stresses for the unmodified functional expressed by Equation (14) are selected independently. With the utilization of the three out-of-plane displacement terms given in Equation (23), the bending stress distributions, in the form of the curvatures as specified by Equation (15), satisfying the boundary conditions [Equation (37)], are

$$\left.\begin{aligned}
\frac{\sigma_x''}{E} &= \left(A_{11}\cos\frac{\pi y}{b} + A_{13}\cos\frac{2\pi y}{b}\right)\cos\frac{\pi x}{\lambda} + A_{31}\cos\frac{3\pi x}{\lambda}\cos\frac{\pi y}{b} \\
\frac{\sigma_y''}{E} &= \left(B_{11}\cos\frac{\pi y}{b} + B_{13}\cos\frac{3\pi y}{b}\right)\cos\frac{\pi x}{\lambda} + B_{31}\cos\frac{3\pi x}{\lambda}\cos\frac{\pi y}{b} \\
\frac{\tau_{xy}''}{E} &= \left(C_{11}\sin\frac{\pi y}{b} + C_{13}\sin\frac{3\pi y}{b}\right)\sin\frac{\pi x}{\lambda} + C_{31}\sin\frac{3\pi x}{\lambda}\sin\frac{3\pi y}{y}
\end{aligned}\right\} \quad (24)$$

where all amplitude parameters are free; however, it is found for rectangular plates that the number of free coefficients may be reduced without significant loss of accuracy by setting $A_{13}/A_{11} = B_{13}/B_{11} = C_{13}/C_{11} = \alpha$ and $A_{31} = B_{31} = C_{31} = 0$. Upon substitution of the expressions of Equations (22), (23), and (24), the modified Reissner functional given by Equation (14) is expanded to yield Equation (42) of Appendix III. Solutions obtained on the basis of the functional, Equation (42), with w given by Equation (23) are designated Solutions B.

INELASTIC PROBLEM, UNMODIFIED REISSNER FUNCTIONAL

For the determination of the inelastic postbuckling behavior and maximum strength of plates, the Reissner function given by Equation (10) is employed in conjunction with the stress distribution described by Equations (22) and (24) and the displacement functions of Equation (23). However, due to the Ramberg-Osgood generalized stress-strain curve parameter n being large for conventional metals in general, the indicated integration in closed form of the effective stress terms in Equation (10) is prohibitive. Consequently, the integration is performed numerically.

For $K = 0$, Equation (10) reduces to the functional given by Equation (42). Hence, the functional for the inelastic problem may be written as

$$\frac{U''}{EV} = \left(\frac{U''}{EV}\right)_{elastic} - \frac{1}{2}\frac{K}{n+1}\int_0^1\int_0^1 \left|\left(\frac{\sigma_{eff}}{E}\right)_t^{n+1} + \left(\frac{\sigma_{eff}}{E}\right)_b^{n+1}\right| d\xi d\eta \tag{25}$$

where $(U''/EV)_{elastic}$ is given in Appendix III by Equation (42).

The required solutions are obtained by determining the values of the stress and displacement coefficients that render the formulated Reissner functionals stationary in value. To achieve elastic Solutions A1, A2, and B and the inelastic solution, the procedure requires the solutions of 8, 9, 13, and 13 simultaneous nonlinear algebraic equations respectively. The method utilized in the solution of these equations is a basic Newton-Raphson iteration procedure. Appendix IV contains a detailed description of the method. The modified steepest descents minimization technique developed in Reference 1 and improved in Reference 2 to obtain relative minima of the strain-energy functional is not as efficient for the present solutions, since the vanishing of the first variation of the Reissner or modified-Reissner functionals with respect to purely arbitrary stress and displacement states does not correspond to the achievement of relative minima.

The application of the Newton-Raphson technique yields numerical values for the coefficients which can be utilized in conjunction with the axial stress distribution to construct load-shortening curves. The load-shortening curves are plots of average axial stress versus unit shortening, normalized with respect to the buckling stress and strain for a square (or infinitely long), simply supported plate, respectively. The average stresses in the present analyses are obtained directly from an integration of σ_x' over the plate width. Numerical values of the stress and displacement coefficients, normalized with respect to the critical strain of a simply supported square (or infinitely long plate), are listed in Tables I through VIII for various values of unit shortening.

Two solutions, denoted by Solution A1 and Solution A2, are effected for square plates employing the modified elastic functional and are compared in Figure 3 to the first-approximation elastic solution of Reference 2 and the essentially "exact" solutions of References 3 and 5; corresponding normalized stress and displacement coefficients for specific values of unit shortening are listed in Tables I and II. Solution A1, which utilizes only the w_{11} and w_{31} terms in the w-displacement function, is accurate only for square plates. Analyses results for rectangular plates with aspect ratios 1.25 , 1.33 , 1.5 , and 2.0 , obtained from utilization of the modified (Solutions A2) and unmodified (Solutions B) elastic functionals, are compared in Figures 4 through 7 to the first-approximation elastic solutions of Reference 2 and the essentially "exact" solutions of Reference 3. Tables II through V contain representative values of the normalized stress and displacement coefficients, at given unit shortenings, for these cases.

A maximum strength analysis of an aluminum 2024-T3 rectangular plate with aspect ratio 5 , for which test data exist in Reference 3, is obtained through utilization of the unmodified inelastic functional. The two-element plate geometry analyzed is made effectively equivalent to the plate tested in Reference 3 by equating the expression for the critical stress of the two-element plate, derived in Reference 2 to be $\sigma_{cr} = 4/3[E(\pi h/b)^2]$, to the classical critical stress expression $\sigma_{cr} = E(\pi t/b)^2/3(1-\mu^2)$ for a solid plate. The load-shortening curve obtained from this analysis is compared in Figure 8 to the test data of Reference 3 and the analysis results of Reference 2. Tables VI through VIII contain numerical values for the elastic and inelastic stress and displacement solution coefficients, at various values of unit shortening, corresponding to the buckle wavelengths considered. The determination of the values of unit-shortening ratios at which the buckle wavelengths change as the unit shortening increases is described in detail in Reference 2 and is briefly restated herein for completeness. The calculation is made by construction of load-shortening curves corresponding to each buckle half wavelength λ or the number of buckles m .

For a given plate geometry, the load-shortening curves corresponding to m and m+1 buckles intersect. At a particular value of shortening, the Reissner functional magnitude (area under the load-shortening curve) corresponding to m+1 buckles will be equal to that corresponding to m buckles. Theoretically, this value of unit shortening is the point at which a "jump" is first possible. For additional shortening, the curve corresponding to m+1 buckles becomes temporarily the appropriate load-shortening path. The procedure is followed with successively increased numbers of buckles considered to the point at which a maximum load is reached.

RESULTS AND DISCUSSION

ELASTIC SOLUTIONS

To establish confidence in the method of analysis employed herein, applications of Reissner's principle and a modified version of the principle have been made for comparison with essentially "exact" solutions to the elastic postbuckling problem of uniaxially compressed, simply supported plates possessing straight, unloaded edges. Excellent agreement of the present results with the essentially "exact" solutions of Stein (Reference 3) and Levy (Reference 5), shown in Figures 3 through 7, indicates that the method of solution is sufficiently flexible to permit appreciable simplifications to be made without significant loss of accuracy.

One simplification, that of coupling the bending stresses to the curvatures, introduced in Reference 9 and employed herein to perform Solutions A, is seen to be a desirable procedure for elastic plate-type problems, in that the number of free stress parameters in the direct variational solution is considerably reduced. Another simplification, that of describing the u-displacement function by a single term and the specification of $v = 0$ (utilized in both Solutions A and B), which allows the in-plane equilibrium equations to be satisfied identically, appears to be justified by the excellent agreement obtained upon comparison with the essentially "exact" solutions. Obviously, if the stresses were required to be found in terms of the displacements through application of the constitutive law (potential energy approach), the influence of the simplified u- and v-displacement functions would have a profound effect, relative to the satisfaction of in-plane equilibrium. Hence, the feature of selecting the stresses and displacements independently and of satisfying Hooke's law only approximately, as performed herein for Solutions B with the aid of Reissner's variational principle, offers a very desirable approach relative to that of potential energy, wherein very accurate displacement distributions are required to attain satisfactory stress distributions through the constitutive law.

The reduced complexity of the problem, as a consequence of the invoked simplifications, is strikingly illustrated by the fact that Levy (Reference 5) requires at least four out-of-plane displacements and fourteen stress function terms to obtain the essentially "exact" solution shown in Figure 3 for the square plate at the maximum shortening ratio considered. At the shortening ratio corresponding to the calculation limit of Stein's solution (Reference 3), at least three out-of-plane displacement terms and either ten stress-function terms or ten in-plane-displacement terms are required to effect the "exact" solution by the Levy and Stein approaches, respectively.

INELASTIC SOLUTION

The maximum strength analysis of a uniformly shortened, simply supported rectangular plate based on Reissner's variational principle has been performed to justify the procedure for adjusting, in a gross manner, the load-shortening curve to account for waveform changes beyond initial buckling. This procedure was utilized in the solutions of References 1 and 2 in conjunction with the potential energy method and with the assumed persistence during postbuckling of the waveform occurring at initial buckling. The load-shortening curve obtained in the present analysis for the particular plate geometry reflected in the experimental work of Reference 3 agrees well with the results of Reference 2, as shown in Figure 8. However, both the present results and the results of Reference 2 are slightly conservative with respect to the experimental data of Reference 3; the increased slope of the experimental curve at initial buckling indicates that the test data have been influenced to some degree by the test fixtures.

To place the present maximum strength solution in more general perspective, it is plotted in Figure 9 along with a portion of the theoretical and experimental plate maximum strength data compiled in Figure 5 of Reference 2. The reference numbers in Figure 9 are those of the present report. It can be seen that the maximum strength analyses of both References 2 and 3 correlate more closely with the test data of Reference 11 than with the test point taken from Reference 3. However, as mentioned previously, the test data of Reference 3 reflect an apparent increase in plate strength due to test fixture effects; however, since the actual cause of the increased slope of the load-shortening curve at initial buckling (see Figure 8) could not be isolated, no attempt has been made to adjust the experimental postbuckling and maximum strength results. The comparison of the present results and those of Reference 2 with the maximum strength criterion based upon the von Kármán effective-width concept (Reference 10) suggests that when the buckling stress for a simply supported plate is greater than $0.3\ \sigma_{cy}$, the curve faired through the three points given by the calculated point of the present analysis and the calculated points of Reference 2 at $(\sigma_{cr})_{cl}/\sigma_{cy} = 0.3$ and 0.6, respectively, should be utilized for maximum strength prediction of rectangular plates $(L/b > 5)$. Application of the present analysis technique to improve the results of Reference 2 at $(\sigma_{cr})_{cl}/\sigma_{cy} = 0.3$ and 0.6, respectively, would show little effect, since the higher initial buckling stresses reduce the postbuckling range considerably prior to the occurrence of a maximum load. In fact, a glance at Figure 5 of Reference 2 shows that the maximum strengths of simply supported plates with various types of unloaded-edge conditions after buckling reflect little difference as $(\sigma_{cr})_{cl}/\sigma_{cy}$ approaches about 0.6.

The complexity of extending the potential energy (Rayleigh-Ritz) solution of Reference 2 (without the load-shortening curve adjustment procedure) to obtain results equivalent to those of the present analysis is prohibitive, due to the number of displacement terms required to obtain an accurate in-plane stress distribution through the constitutive law. The procedure employed in References 1 and 2 requires the existence of an "exact" elastic solution upon which to base the adjustment procedure; on the other hand, efficient use of the present theory requires, in the absence of stress distribution information determined on the basis of limited experimental data, no more than approximate elastic solutions from which forms of the assumed stress distributions can be obtained.

Another advantage of the present method of analysis relative to the potential energy approach of References 1 and 2 is that in the former, the stress distribution can be evaluated throughout the plate after the stationary value of the Reissner functional has been determined, whereas in the latter, the stress distribution is known only for the solution based upon the initial buckling waveform. The stress distributions corresponding to the adjusted load-shortening curves remain, of course, unknown.

Due to the utilization of a deformation theory of plasticity, occurrence of unloading in particular regions of the plate as the shortening and number of buckles both increase must be considered. It is found in the present analysis that local unloading of small magnitude (no greater than 5%) does occur in inelastic regions of the plate when the buckle wavelength is assumed to change abruptly; however, due to the known inaccuracy of the uniaxial stress-strain curve representation for a material and taking cognizance of the fact that the total load carried by the plate is an integrated phenomenon, the small, local unloading that does occur in the inelastic portion of the plate is ignored in the present problem. Thus, it may be considered that the present problem has been undertaken under the assumption that the material is simply of a nonlinear elastic nature.

The simplifying assumption in the present solution that the v-displacement is zero throughout the plate should be assessed. Based on the correlation of the results for the elastic solutions carried out herein and in References 3 and 5, the influence of the violation of the condition that the unloaded edges be free to translate is considered to be minor. The actual correlation is shown in Figures 3 through 7.

CONCLUDING REMARKS

The basic factors required to accurately predict the inelastic postbuckling behavior and the maximum strength of simply supported rectangular plates with straight, unloaded edges have been shown to be (1) buckle wavelength changes, (2) changes in waveform beyond initial buckling, and (3) the onset of plasticity. The impact of item (1) can be seen in the inelastic postbuckling analysis of simply supported square plates presented in Reference 1. While the buckle aspect ratio remains unity, no clearly defined plate maximum strength is obtained for the quite reasonable range of plate end shortening considered. However, the consideration of all three factors collectively for the first time in the rectangular plate investigation of Reference 2 results in an analytical solution for the prediction of the maximum strength of a simply supported plate, which correlates excellently with experimental data presented in Reference 3. The experiment reflects loading and boundary conditions very close to those assumed in analyses of References 1 and 2. With regard to item (2), the analyses of References 1 and 2 utilize a load-shortening curve adjustment procedure that only grossly accounts for the waveform changes beyond initial buckling as they effect the load-shortening curve. The present analysis, based upon Reissner's variational principle, establishes the validity of the adjustment procedure and, in addition, illustrates another attractive method for obtaining engineering solutions to plate maximum strength problems. As concerns item (3), the ability of the secant modulus deformation theory of plasticity to adequately represent inelastic behavior is demonstrated. Local unloading of small magnitudes occurs in certain inelastic portions of the plates analyzed; however, the influence of small unloading on the postbuckling and maximum plate strengths (which are integrated phenomena) is not significant.

Efficient use of the potential energy method of solution with the adjustment procedure incorporated requires that essentially "exact" elastic solutions be available to provide the necessary basis for carrying out the adjustment procedure. However, should neither "exact" elastic solutions exist nor limited experimental data be available to suggest the form of the stress distributions, then only approximate elastic solutions are required to obtain the general form of the distributions to be used in conjunction with Reissner's variational principle.

The fact that Reissner's principle has had little application to date may be attributed to the apparent complexity involved in the requirement to assume the states of stress and displacement independently. However, the present analyses indicate that the principle has sufficient flexibility to permit major simplifications to be made with little loss in accuracy. In particular, the features of (1) employing only the first approximation of the in-plane displacements and (2) coupling of the bending stresses to the curvatures through the constitutive law in the elastic case (the modified Reissner principle introduced in Reference 9) are found to be extremely useful in reducing the number of free amplitude parameters appearing in the assumed stress and displacement functions. Item (1), although resulting in a violation of the unloaded-edges boundary condition and permitting only approximate satisfaction of the constitutive law, is seen to have only minor influence on the accuracy of the elastic solutions. The inference of item (2) is also found effective for the present inelastic solution, where the bending stresses are assumed in the <u>form</u> of the curvatures as indicated by the constitutive law but with completely free amplitude coefficients whose magnitudes are determined from the variational process.

The unconservative nature of the von Kármán effective-width formula, when applied to simply supported plates possessing a buckling stress greater than $0.3\ (\sigma_{cy})$, suggests that the combined results of the present inelastic analysis and the inelastic analyses of Reference 2 be utilized for design purposes. However, additional tests, utilizing the technique described in Reference 3, are required for the range $0.3 < (\sigma_{cr})_{cl}/\sigma_{cy} < 0.6$. For simply supported plates with buckling stresses less than $0.3\ (\sigma_{cy})$, the von Kármán effective-width formula provides conservative design data for plates in which the edges do not move out of the original plane in the postbuckling range. Where other plate loading and boundary conditions are of interest, the techniques of either Reference 2 or the present inelastic analysis provide an efficient means for obtaining maximum strength behavior. The combination of the results of such analyses with a minimum of judicious testing (for correlation purposes) could provide a useful tool for removing some of the conservatism from present design analysis techniques.

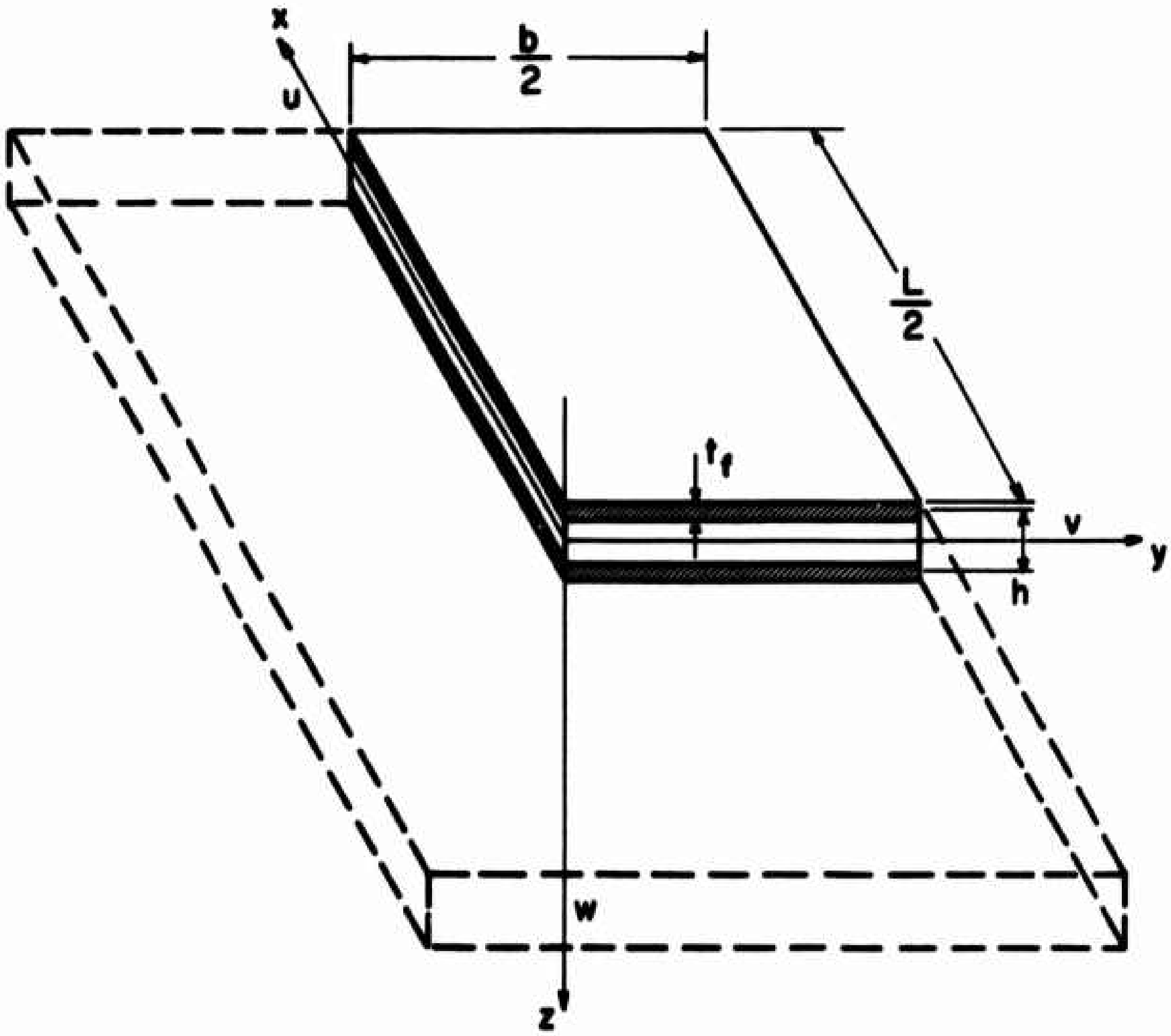

Figure 1. Plate Geometry and Coordinate System.

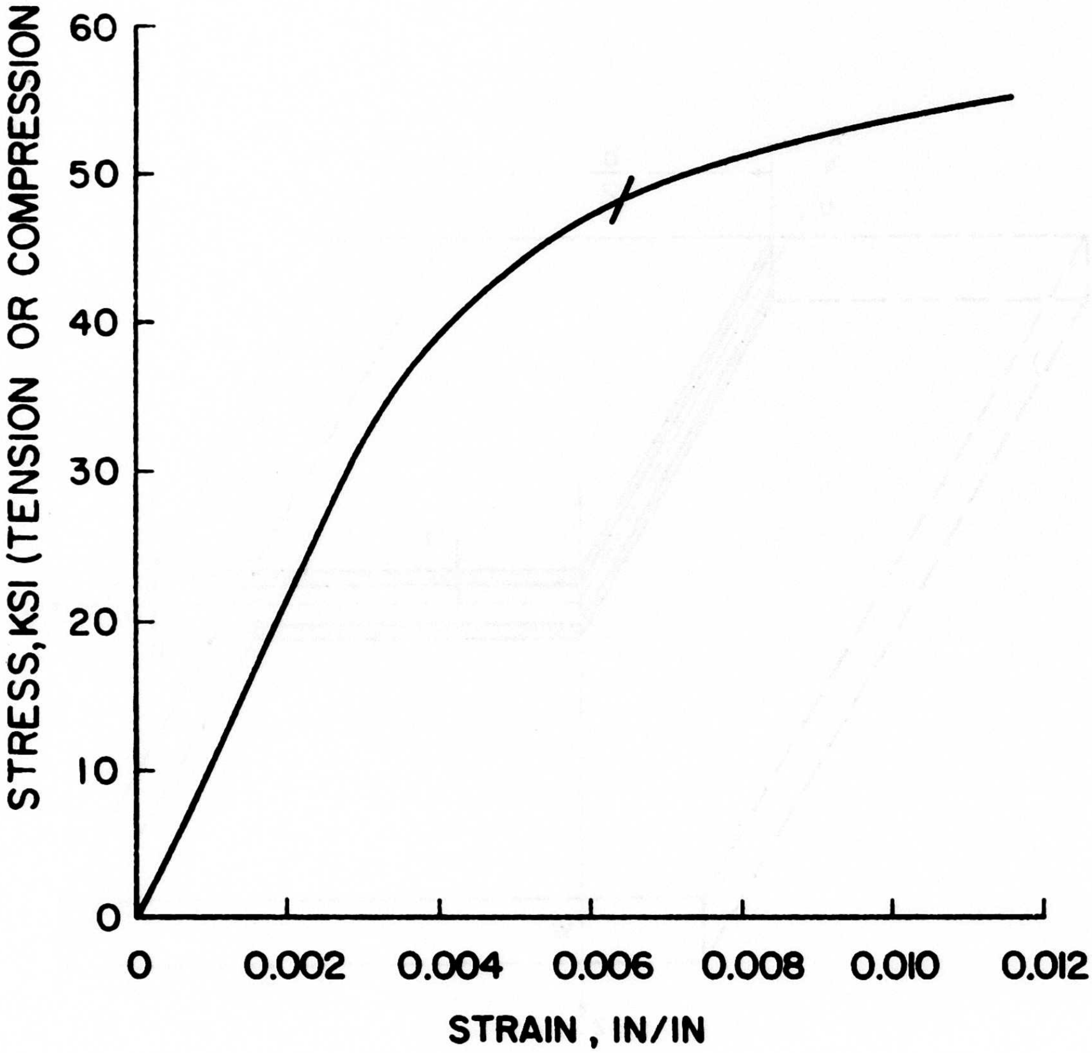

Figure 2. Stress-Strain Curve for Aluminum 2024-T3 Assumed in Present Inelastic Analyses.

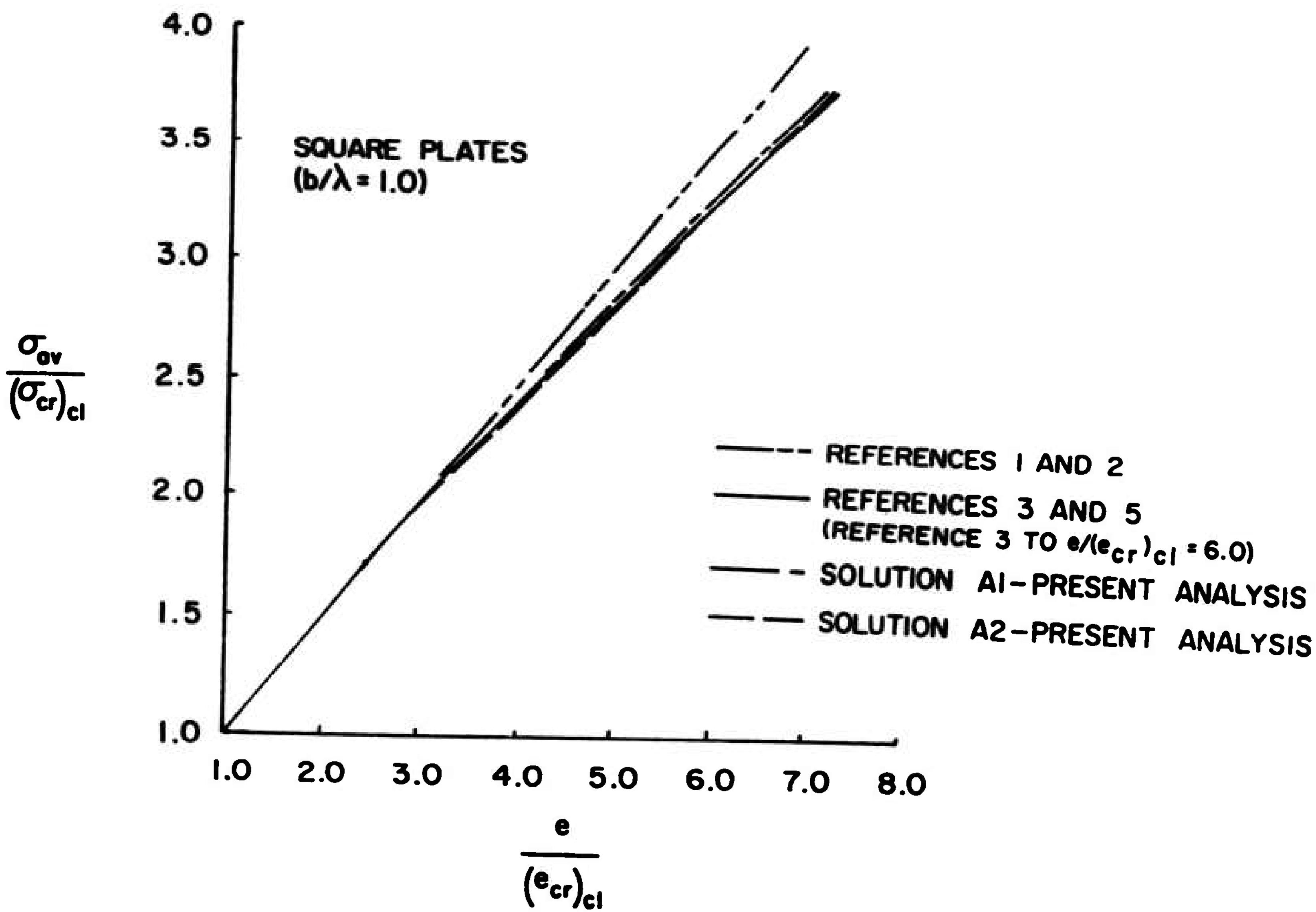

Figure 3. Load-Shortening Curves for Square Plates Based on Elastic Solutions.

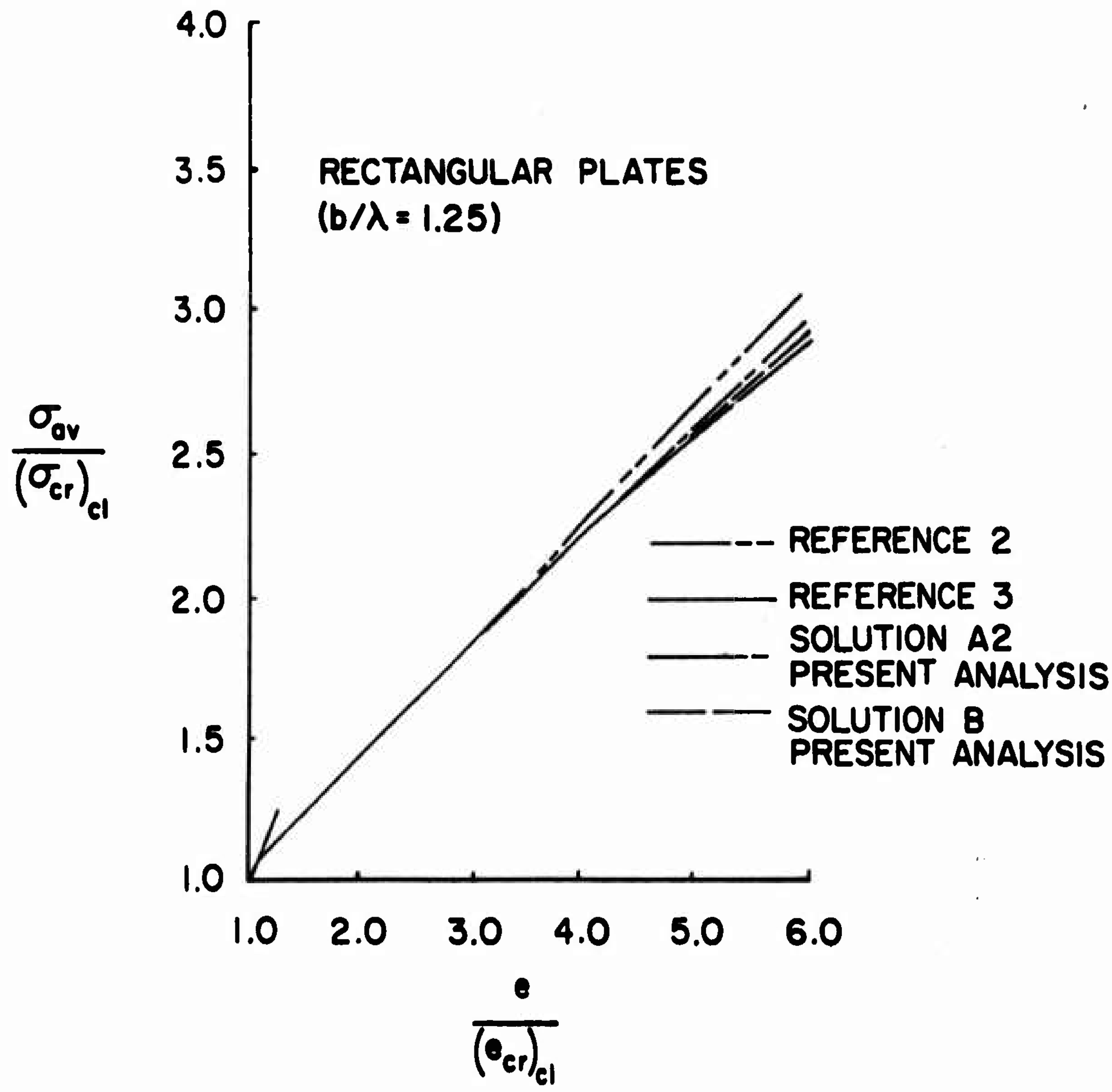

Figure 4. Load-Shortening Curves for Rectangular Plates Based on Elastic Solutions $(b/\lambda = 1.25)$.

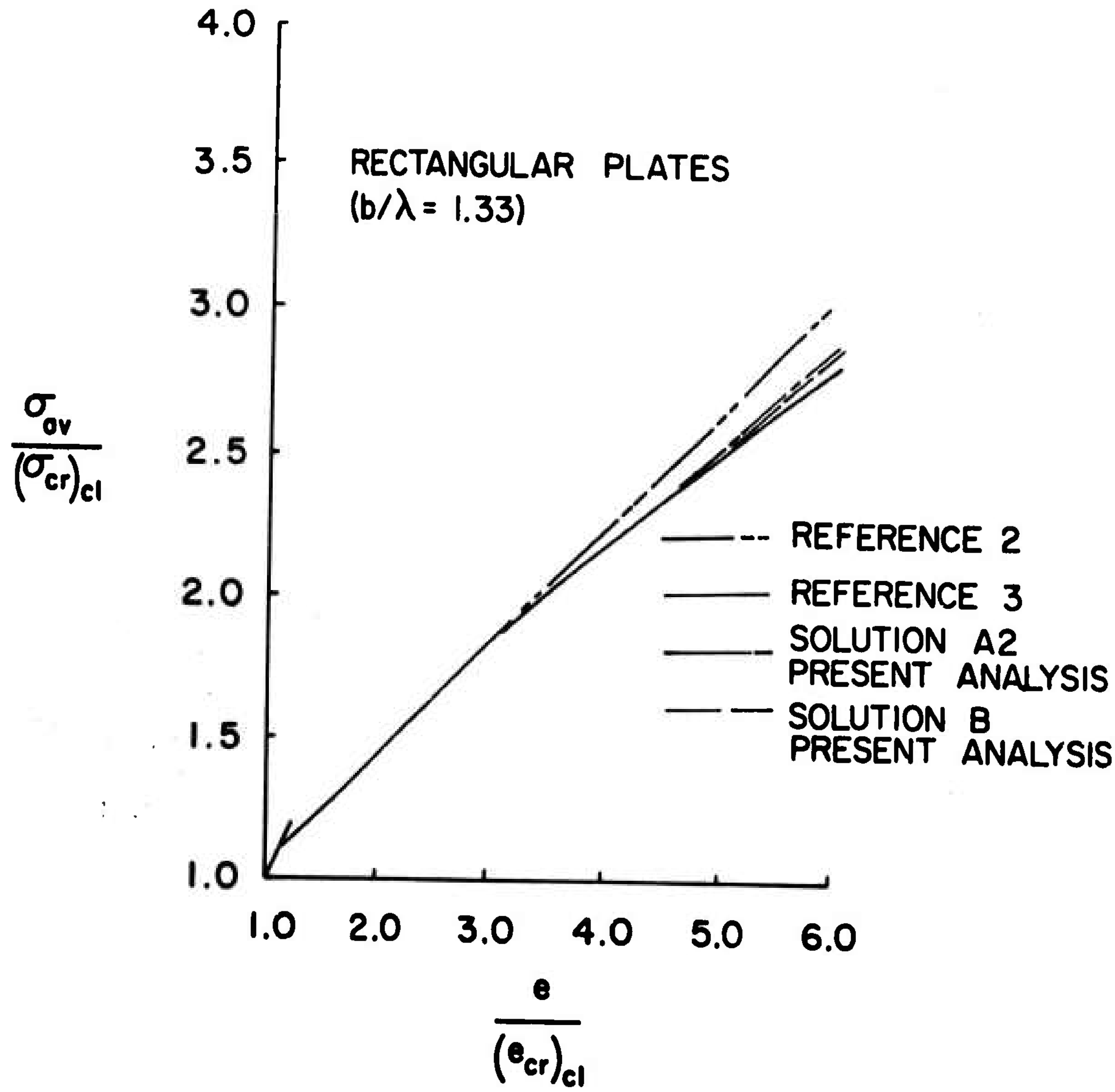

Figure 5. Load-Shortening Curves for Rectangular Plates Based on Elastic Solutions $(b/\lambda = 1.33)$.

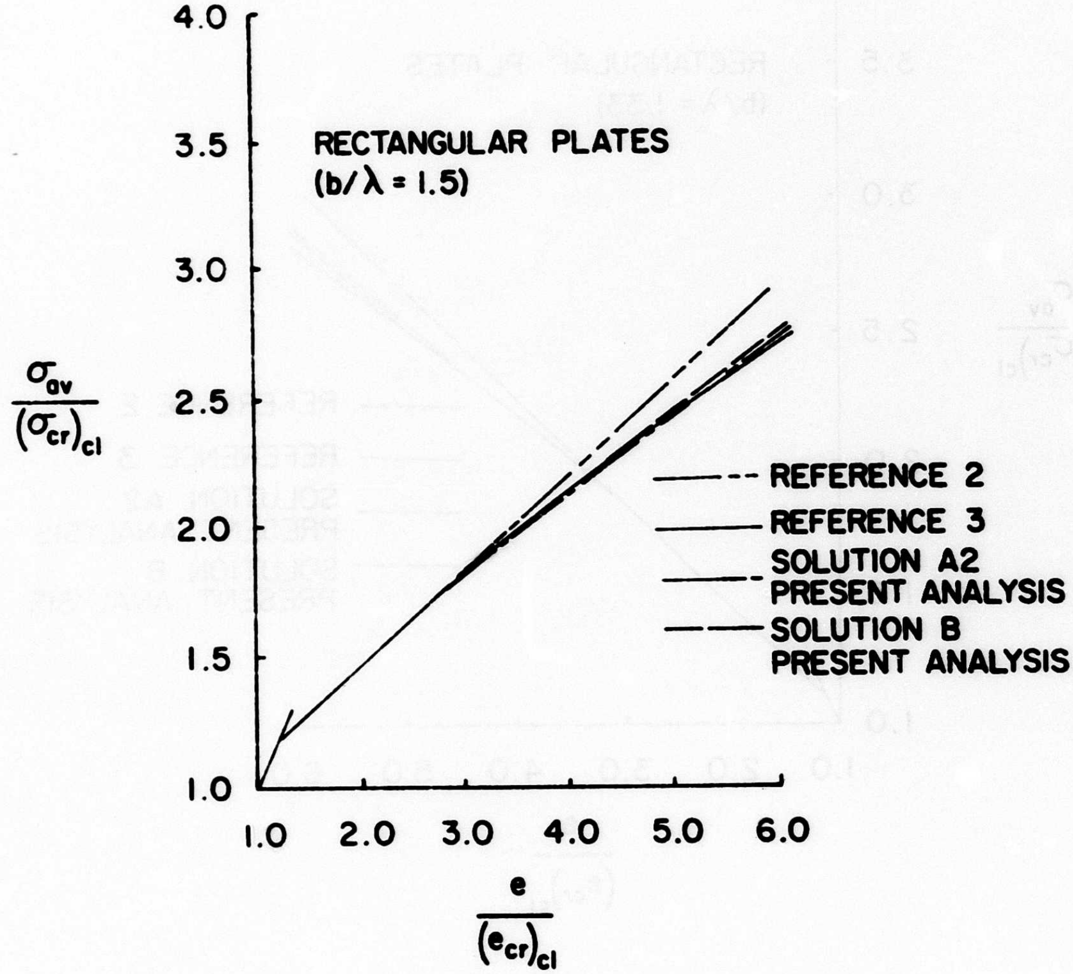

Figure 6. Load-Shortening Curves for Rectangular Plates Based on Elastic Solutions ($b/\lambda = 1.5$) .

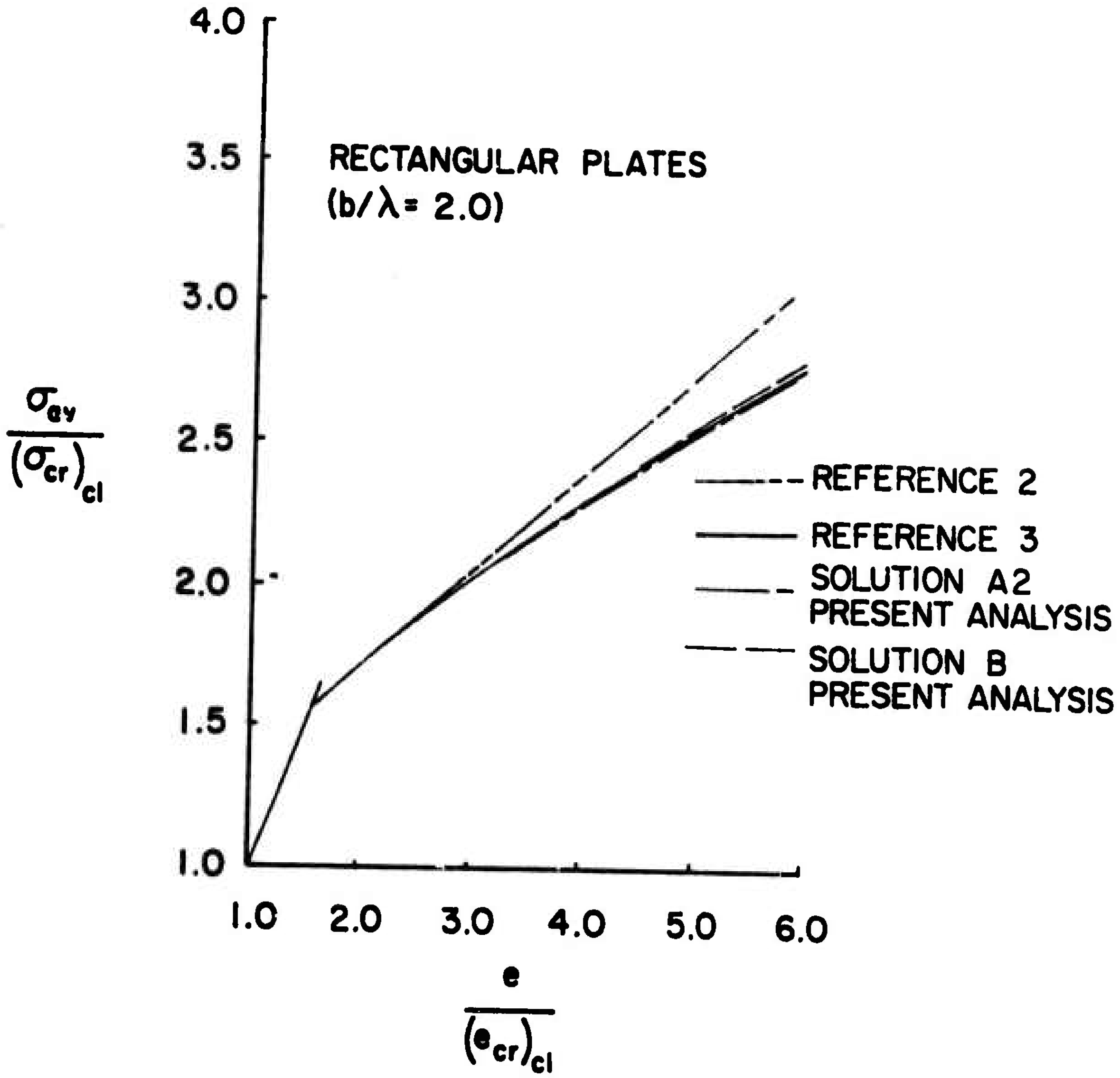

Figure 7. Load-Shortening Curves for Rectangular Plates Based on Elastic Solutions $(b/\lambda = 2.0)$.

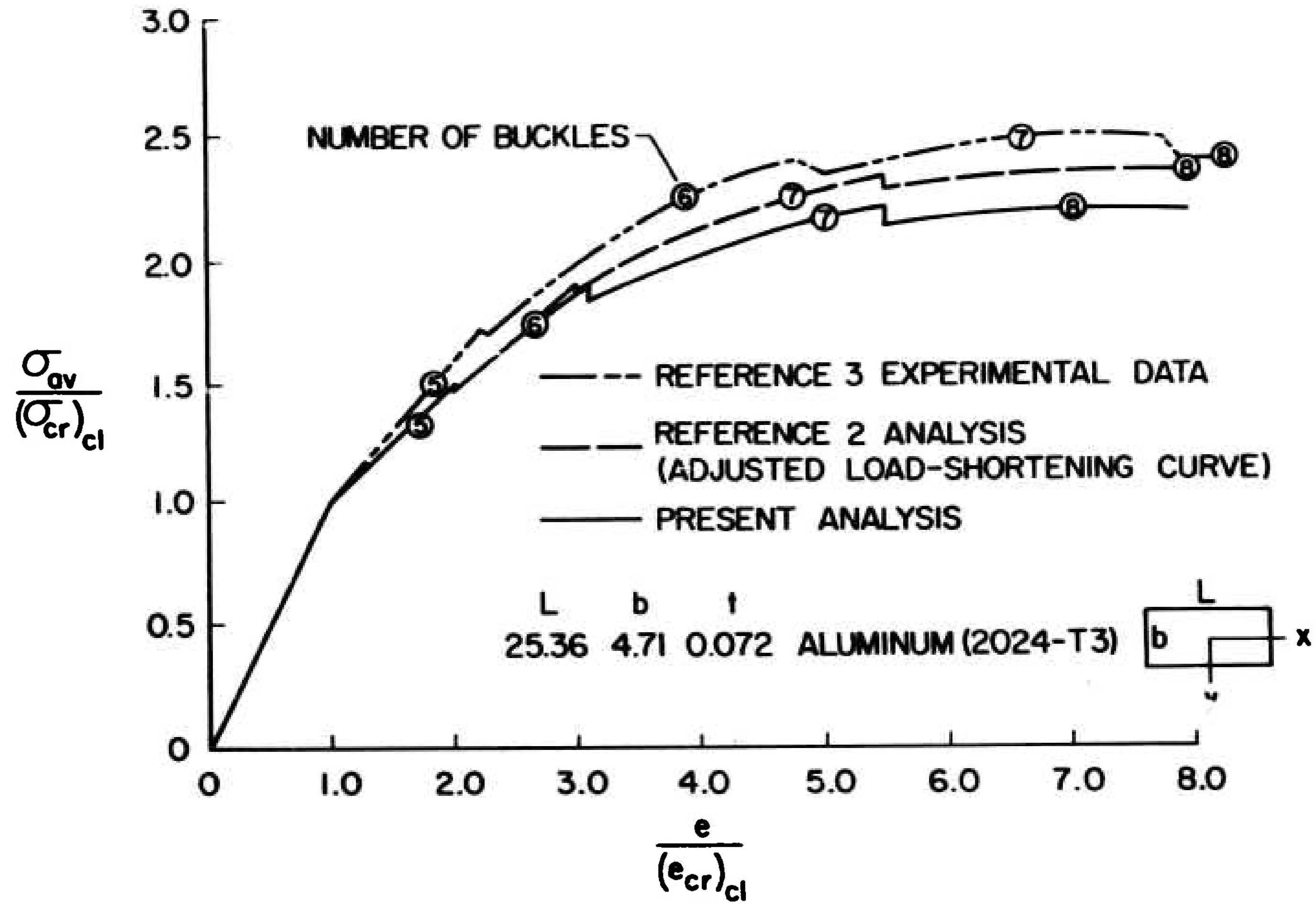

Figure 8. Comparison of Load-Shortening Curves for a Rectangular Plate Based on Prediction of Present Theory, Analysis of Reference 2, and Experimental Results of Reference 3.

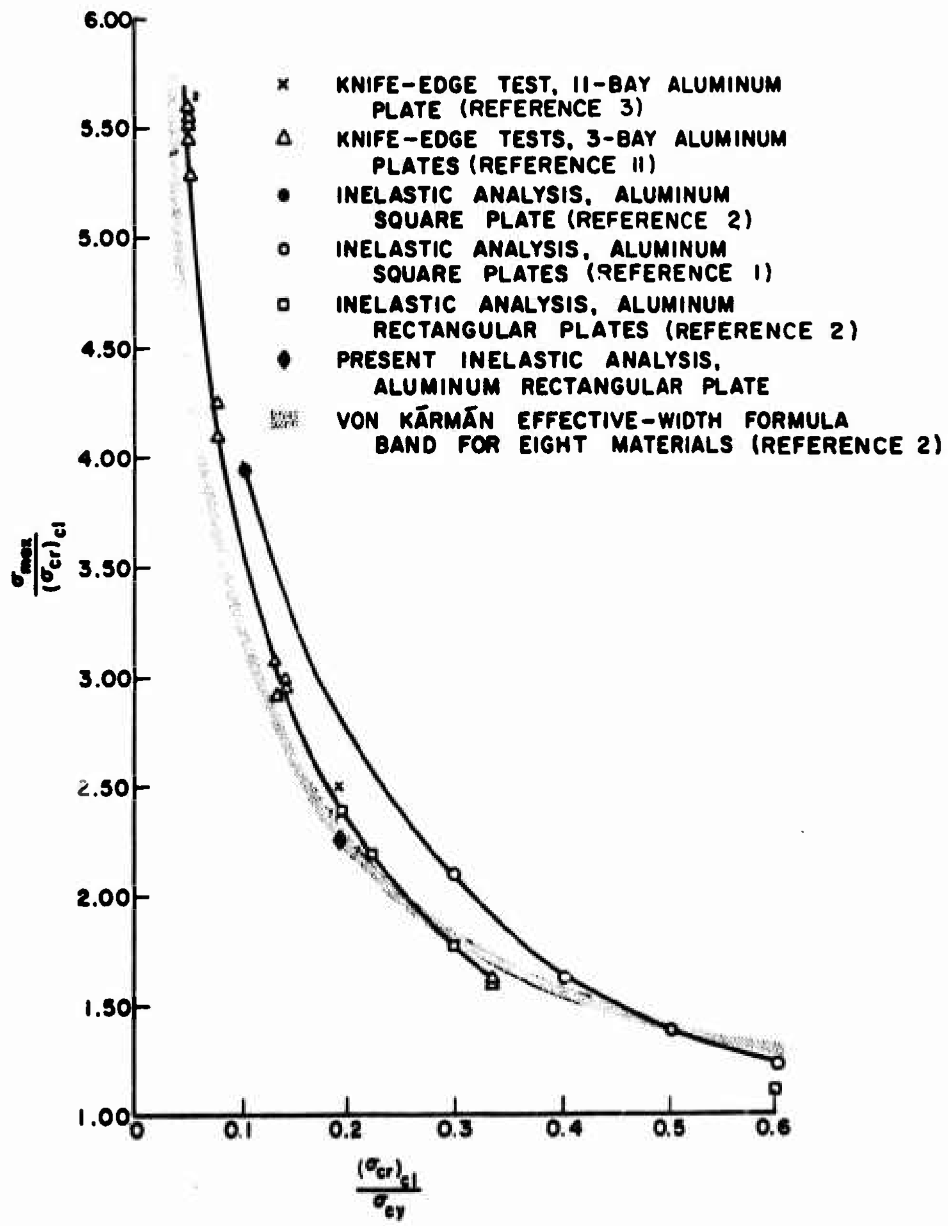

Figure 9. Theoretical and Empirical Maximum Strength Criteria for Simply Supported Rectangular Plates.

TABLE I. NORMALIZED STRESS AND DISPLACEMENT COEFFICIENTS FOR SQUARE PLATES, SOLUTION A1

$\frac{e}{(e_{cr})_{cl}}$	$\frac{c_{av}}{(\sigma_{cr})_{cl}}$	$\frac{A_1}{(e_{cr})_{cl}}$	$\frac{A_2}{(e_{cr})_{cl}}$	$\frac{A_3}{(e_{cr})_{cl}}$	$\frac{B_2}{(e_{cr})_{cl}}$	$\frac{B_3}{(e_{cr})_{cl}}$	$\frac{C}{(e_{cr})_{cl}}$	$\frac{\bar{w}_{11}}{(e_{cr})_{cl}}$	$\frac{\bar{w}_{31}}{(e_{cr})_{cl}} \times 10^2$
2.0	- 1.494	- 0.998	- 2.317	1.383	- 2.073	1.122	0.239	0.554	- 0.912
2.5	- 1.735	- 0.987	- 3.545	2.165	- 2.972	1.555	0.555	0.680	- 1.728
3.0	- 1.972	- 0.967	- 4.823	3.012	- 3.765	1.890	1.011	0.785	- 2.726
3.5	- 2.206	- 0.939	- 6.154	3.925	- 4.444	2.124	1.611	0.879	- 3.881
4.0	- 2.435	- 0.903	- 7.536	4.903	- 5.009	2.254	2.349	0.963	- 5.168
4.5	- 2.659	- 0.859	- 8.966	5.941	- 5.460	2.286	3.218	1.039	- 6.560
5.0	- 2.880	- 0.807	-10.442	7.036	- 5.801	2.223	4.205	1.109	- 8.030
5.5	- 3.098	- 0.748	-11.957	8.179	- 6.041	2.075	5.295	1.174	- 9.556
6.0	- 3.312	- 0.683	-13.508	9.367	- 6.188	1.848	6.476	1.234	-11.118

TABLE II. NORMALIZED STRESS AND DISPLACEMENT COEFFICIENTS FOR SQUARE AND RECTANGULAR PLATES, SOLUTIONS A2

$\frac{e}{(e_{cr})_{cl}}$	$\frac{\sigma_{av}}{(\sigma_{cr})_{cl}}$	$\frac{A_1}{(e_{cr})_{cl}}$	$\frac{A_2}{(e_{cr})_{cl}}$	$\frac{A_3}{(e_{cr})_{cl}}$	$\frac{B_2}{(e_{cr})_{cl}}$	$\frac{B_3}{(e_{cr})_{cl}}$	$\frac{C}{(e_{cr})_{cl}}$	$\frac{\bar{w}_{11}}{(e_{cr})_{cl}}$	$\frac{\bar{w}_{31}}{(e_{cr})_{cl}} \cdot 10^2$	$\frac{\bar{w}_{13}}{(e_{cr})_{cl}} \cdot 10^2$
$\beta = 1.00$										
2.0	- 1.490	- 1.004	- 2.264	1.346	- 2.083	1.132	0.357	0.555	- 0.908	- 0.505
2.5	- 1.728	- 1.001	- 3.431	2.083	- 2.991	1.576	0.805	0.683	- 1.717	- 0.879
3.0	- 1.961	- 0.993	- 4.628	2.868	- 3.803	1.926	1.430	0.790	- 2.708	- 1.288
3.5	- 2.189	- 0.978	- 5.858	3.709	- 4.505	2.177	2.226	0.885	- 3.854	- 1.718
4.0	- 2.412	- 0.958	- 7.123	4.600	- 5.093	2.328	3.189	0.969	- 5.132	- 2.161
4.5	- 2.631	- 0.932	- 8.422	5.539	- 5.570	2.380	4.301	1.047	- 6.518	- 2.614
5.0	- 2.845	- 0.899	- 9.752	6.523	- 5.940	2.340	5.551	1.118	- 7.987	- 3.073
5.5	- 3.055	- 0.861	-11.110	7.547	- 6.207	2.211	6.923	1.185	- 9.516	- 3.540
6.0	- 3.262	- 0.819	-12.491	8.605	- 6.381	2.003	8.402	1.246	-11.088	- 4.013
$\beta = 1.25$										
2.0	- 1.435	- 0.908	- 2.221	1.068	- 1.620	0.887	- 0.498	0.469	- 0.232	- 0.744
3.0	- 1.824	- 0.774	- 4.363	2.022	- 3.282	1.778	- 0.718	0.676	- 0.688	- 2.010
4.0	- 2.201	- 0.654	- 6.348	2.845	- 4.892	2.617	- 0.670	0.835	- 1.289	- 3.444
5.0	- 2.568	- 0.544	- 8.219	3.581	- 6.442	3.397	- 0.387	0.970	- 2.009	- 4.931
6.0	- 2.927	- 0.443	-10.010	4.260	- 7.923	4.113	0.107	1.088	- 2.832	- 6.417
$\beta = 1.333$										
2.0	- 1.440	- 0.920	- 2.167	1.013	- 1.440	0.783	- 0.563	0.437	- 0.152	- 0.766
3.0	- 1.810	- 0.761	- 4.269	1.869	- 3.003	1.625	- 0.942	0.637	- 0.462	- 2.141
4.0	- 2.167	- 0.621	- 6.147	2.518	- 4.554	2.448	- 1.114	0.790	- 0.871	- 3.714
5.0	- 2.513	- 0.498	- 7.853	3.011	- 6.087	3.247	- 1.110	0.920	- 1.362	- 5.351
6.0	- 2.852	- 0.387	- 9.429	3.391	- 7.593	4.017	- 0.953	1.034	- 1.924	- 6.989

TABLE III. NORMALIZED STRESS AND DISPLACEMENT COEFFICIENTS FOR RECTANGULAR PLATES, SOLUTIONS A2

$\frac{e}{(e_{cr})_{cl}}$	$\frac{\sigma_{av}}{(\sigma_{cr})_{cl}}$	$\frac{A_1}{(e'_{cr})_{cl}}$	$\frac{A_2}{(e_{cr})_{cl}}$	$\frac{A_3}{(e_{cr})_{cl}}$	$\frac{B_2}{(e_{cr})_{cl}}$	$\frac{B_3}{(e_{cr})_{cl}}$	$\frac{C}{(e_{cr})_{cl}}$	$\frac{\bar{w}_{11}}{(e_{cr})_{cl}}$	$\frac{\bar{w}_{31}}{(e_{cr})_{cl}} \times 10^2$	$\frac{\bar{w}_{13}}{(e_{cr})_{cl}} \times 10^2$
β = 1.50										
2.0	- 1.475	- 0.991	- 2.006	0.923	- 1.100	0.588	- 0.541	0.376	- 0.066	- 0.739
3.0	- 1.818	- 0.797	- 4.058	1.659	- 2.458	1.316	- 1.054	0.565	- 0.217	- 2.258
4.0	- 2.142	- 0.631	- 5.763	2.050	- 3.851	2.061	- 1.446	0.707	- 0.420	- 4.031
5.0	- 2.454	- 0.490	- 7.194	2.168	- 5.272	2.819	- 1.742	0.827	- 0.665	- 5.890
6.0	- 2.757	- 0.369	- 8.411	2.079	- 6.716	3.583	- 1.959	0.932	- 0.949	- 7.751
β = 2.00										
2.0	- 1.710	- 1.440	- 1.155	0.574	- 0.363	0.184	- 0.188	0.210	- 0.004	- 0.329
3.0	- 2.020	- 1.196	- 3.273	1.334	- 1.234	0.632	- 0.591	0.385	- 0.027	- 1.836
4.0	- 2.298	- 0.999	- 4.748	1.420	- 2.178	1.122	- 0.981	0.507	- 0.061	- 3.759
5.0	- 2.554	- 0.845	- 5.709	0.973	- 3.195	1.651	- 1.373	0.607	- 0.104	- 5.814
6.0	- 2.795	- 0.727	- 6.280	0.126	- 4.275	2.214	- 1.773	0.693	- 0.156	- 7.870

TABLE IV. NORMALIZED STRESS AND DISPLACEMENT COEFFICIENTS FOR RECTANGULAR PLATES, SOLUTIONS B (β = 1.25 and 1.33)

$\frac{e}{(e_{cr})_{cl}}$	$\frac{\sigma_{av}}{(\sigma_{cr})_{cl}}$	$\frac{A_1}{(e_{cr})_{cl}}$	$\frac{A_2}{(e_{cr})_{cl}}$	$\frac{A_3}{(e_{cr})_{cl}}$	$\frac{B_2}{(e_{cr})_{cl}}$	$\frac{B_3}{(e_{cr})_{cl}}$	$\frac{C}{(e_{cr})_{cl}}$	$\frac{\bar{v}_{11}}{(e_{cr})_{cl}}$	$\frac{\bar{w}_{31}}{(e_{cr})_{cl}} \times 10^2$	$\frac{\bar{w}_{13}}{(e_{cr})_{cl}} \times 10^2$	$\frac{A_{11}}{(e_{cr})_{cl}}$	$\frac{B_{11}}{(e_{cr})_{cl}}$	$\frac{C_{11}}{(e_{cr})_{cl}}$	α
β = 1.25														
2.0	- 1.437	- 0.920	- 2.116	0.939	- 1.764	1.014	- 0.689	0.467	0.626	- 0.951	- 2.017	- 1.748	0.611	- 0.074
3.0	- 1.834	- 0.818	- 3.960	1.521	- 3.860	2.289	- 1.501	0.671	1.745	- 2.512	- 2.888	- 2.526	0.875	- 0.137
4.0	- 2.225	- 0.747	- 5.511	1.796	- 6.150	3.733	- 2.405	0.825	3.065	- 4.211	- 3.538	- 3.129	1.073	- 0.188
5.0	- 2.613	- 0.699	- 6.852	1.850	- 8.583	5.303	- 3.387	0.952	4.465	- 5.914	- 4.072	- 3.640	1.235	- 0.229
6.0	- 2.999	- 0.667	- 8.044	1.747	-11.118	6.964	- 4.425	1.063	5.878	- 7.571	- 4.531	- 4.091	1.374	- 0.264
β = 1.33														
2.0	- 1.440	- 0.929	- 2.077	0.906	- 1.542	0.871	- 0.691	0.437	0.467	- 0.969	- 2.082	- 1.734	0.610	- 0.077
3.0	- 1.813	- 0.797	- 3.913	1.434	- 3.441	2.001	- 1.500	0.635	1.374	- 2.693	- 3.016	- 2.540	0.884	- 0.140
4.0	- 2.177	- 0.698	- 5.405	1.610	- 5.534	3.292	- 2.396	0.785	2.495	- 4.557	- 3.713	- 3.166	1.090	- 0.201
5.0	- 2.536	- 0.624	- 6.645	1.515	- 7.784	4.713	- 3.384	0.909	3.735	- 6.425	- 4.284	- 3.699	1.259	- 0.24[illegible]
6.0	- 2.892	- 0.569	- 7.695	1.222	-10.150	6.225	- 4.445	1.015	5.045	- 8.220	- 4.760	- 4.160	1.400	- 0.[illegible]83

TABLE V. NORMALIZED STRESS AND DISPLACEMENT COEFFICIENTS FOR RECTANGULAR PLATES, SOLUTIONS B (β = 1.50 and 2.0)

$\frac{e}{(e_{cr})_{cl}}$	$\frac{\sigma_{av}}{(\sigma_{cr})_{cl}}$	$\frac{A_1}{(e_{cr})_{cl}}$	$\frac{A_2}{(e_{cr})_{cl}}$	$\frac{A_3}{(e_{cr})_{cl}}$	$\frac{B_2}{(e_{cr})_{cl}}$	$\frac{B_3}{(e_{cr})_{cl}}$	$\frac{C}{(e_{cr})_{cl}}$	$\frac{\bar{w}_{11}}{(e_{cr})_{cl}}$	$\frac{\bar{w}_{31}}{(e_{cr})_{cl}} \times 10^2$	$\frac{\bar{w}_{13}}{(e_{cr})_{cl}} \times 10^2$	$\frac{A_{11}}{(e_{cr})_{cl}}$	$\frac{B_{11}}{(e_{cr})_{cl}}$	$\frac{C_{11}}{(e_{cr})_{cl}}$	α
β = 1.50														
2.0	- 1.473	- 0.997	- 1.935	0.842	- 1.151	0.630	- 0.593	0.377	0.247	- 0.970	- 2.169	- 1.684	0.592	- 0.078
3.0	- 1.811	- 0.823	- 3.750	1.307	- 2.711	1.519	- 1.325	0.565	0.812	- 2.870	- 3.241	- 2.549	0.888	- 0.155
4.0	- 2.134	- 0.689	- 5.118	1.302	- 4.450	2.545	- 2.124	0.707	1.569	- 4.972	- 4.033	- 3.222	1.109	- 0.216
5.0	- 2.448	- 0.585	- 6.157	0.948	- 6.351	3.692	- 3.017	0.824	2.472	- 7.080	- 4.681	- 3.798	1.292	- 0.266
6.0	- 2.756	- 0.506	- 6.953	0.337	- 8.389	4.945	- 4.009	0.926	3.479	- 9.129	- 5.236	- 4.309	1.450	- 0.307
β = 2.0														
3.0	- 2.007	- 1.214	- 3.015	1.058	- 1.289	0.669	- 0.618	0.388	0.141	- 2.395	- 3.641	- 2.464	0.817	- 0.142
4.0	- 2.271	- 1.045	- 4.143	0.775	- 2.340	1.229	- 1.078	0.510	0.329	- 4.703	- 4.771	- 3.288	1.082	- 0.213
5.0	- 2.673	- 0.923	- 4.743	- 0.058	- 3.510	1.861	- 1.589	0.610	0.580	- 7.017	- 5.678	- 3.981	1.303	- 0.267
6.0	- 2.749	- 0.835	- 4.960	- 1.267	- 4.780	2.555	- 2.155	0.695	0.889	- 9.210	- 6.450	- 4.590	1.492	- 0.310

TABLE VI. NORMALIZED STRESS AND DISPLACEMENT COEFFICIENTS FOR A RECTANGULAR PLATE, (β = 1.2 , $\pi h/b = 2.45 \times 10^{-2}$); INELASTIC RESULTS BASED ON ALUMINUM 2024-T3 PLATE MATERIAL

$\frac{e}{(e_{cr})_{cl}}$	$\frac{\sigma_{av}}{(\sigma_{cr})_{cl}}$	$\frac{A_1}{(e_{cr})_{cl}}$	$\frac{A_2}{(e_{cr})_{cl}}$	$\frac{A_3}{(e_{cr})_{cl}}$	$\frac{B_2}{(e_{cr})_{cl}}$	$\frac{B_3}{(e_{cr})_{cl}}$	$\frac{C}{(e_{cr})_{cl}}$	$\frac{\bar{w}_{11}}{(e_{cr})_{cl}}$	$\frac{\bar{w}_{31}}{(e_{cr})_{cl}} \times 10^2$	$\frac{\bar{w}_{13}}{(e_{cr})_{cl}} \times 10^2$	$\frac{A_{11}}{(e_{cr})_{cl}}$	$\frac{B_{11}}{(e_{cr})_{cl}}$	$\frac{C_{11}}{(e_{cr})_{cl}}$	α
β = 1.2 Elastic Solution B														
2.0	- 1.441	- 0.923	- 2.128	0.961	- 1.904	1.106	- 0.659	0.485	0.738	- 0.911	- 1.970	- 1.752	0.609	- 0.071
3.0	- 1.854	- 0.846	- 3.973	1.578	- 4.127	2.477	- 1.451	0.692	1.998	- 2.371	- 2.804	- 2.514	0.870	- 0.130
4.0	- 2.265	- 0.797	- 5.557	1.921	- 6.539	4.018	- 2.335	0.849	3.436	- 3.954	- 3.427	- 3.102	1.059	- 0.178
5.0	- 2.672	- 0.768	- 6.967	2.072	- 9.082	5.679	- 3.283	0.787	4.921	- 5.542	- 3.940	- 3.600	1.218	- 0.217
β = 1.2 Inelastic Solution														
3.0	- 1.834	- 0.837	- 4.010	1.679	- 4.152	2.467	- 1.441	0.705	1.932	- 2.362	- 2.895	- 2.598	0.903	0.135

TABLE VII. NORMALIZED STRESS AND DISPLACEMENT COEFFICIENTS FOR A RECTANGULAR PLATE, (β = 1.4 , $\pi h/b$ = 2.545 × 10^{-2}); INELASTIC RESULTS BASED ON ALUMINUM 2024-T3 PLATE MATERIAL

$\frac{e}{(e_{cr})_{cl}}$	$\frac{\sigma_{av}}{(\sigma_{cr})_{cl}}$	$\frac{A_1}{(e_{cr})_{cl}}$	$\frac{A_2}{(e_{cr})_{cl}}$	$\frac{A_3}{(e_{cr})_{cl}}$	$\frac{B_2}{(e_{cr})_{cl}}$	$\frac{B_3}{(e_{cr})_{cl}}$	$\frac{C}{(e_{cr})_{cl}}$	$\frac{\bar{v}_{11}}{(e_{cr})_{cl}}$	$\frac{\bar{v}_{31}}{(e_{cr})_{cl}} \times 10^2$	$\frac{\bar{v}_{13}}{(e_{cr})_{cl}} \times 10^2$	$\frac{A_{11}}{(e_{cr})_{cl}}$	$\frac{B_{11}}{(e_{cr})_{cl}}$	$\frac{C_{11}}{(e_{cr})_{cl}}$	α
β = 1.4 Elastic Solution B														
2.0	- 1.450	- 0.949	- 2.030	0.881	- 1.376	0.767	- 0.663	0.413	0.365	- 0.996	- 2.124	- 1.718	0.605	- 0.079
3.0	- 1.800	- 0.805	- 3.750	1.275	- 3.060	1.770	- 1.400	0.594	1.177	- 2.900	- 2.985	- 2.454	0.852	- 0.143
4.0	- 2.152	- 0.682	- 5.299	1.477	- 5.073	2.969	- 2.314	0.753	2.085	- 4.766	- 3.846	- 3.191	1.100	- 0.208
5.0	- 2.491	- 0.593	- 6.457	1.272	- 7.178	4.275	- 3.275	0.874	3.190	- 6.745	- 4.447	- 3.742	1.274	- 0.255
6.0	- 2.826	- 0.526	- 7.407	0.844	- 9.413	5.686	- 4.328	0.980	4.378	- 8.668	- 4.962	- 4.229	1.425	- 0.295
β = 1.4 Inelastic Solution														
3.0	- 1.783	- 0.794	- 3.793	1.376	- 3.085	1.760	- 1.390	0.607	1.111	- 2.891	- 3.076	- 2.538	0.885	- 0.148
4.0	- 2.018	0.675	- 5.150	1.845	- 4.720	2.750	- 1.780	0.765	2.250	- 5.200	- 3.680	- 3.130	1.092	- 0.192
5.0	- 2.158	- 0.574	- 6.133	2.303	- 6.080	3.461	- 1.884	0.880	2.870	- 7.605	- 4.034	- 3.631	1.201	- 0.217
6.0	- 2.243	- 0.498	- 6.905	2.782	- 7.116	3.946	- 1.813	0.989	3.717	- 9.879	- 4.253	- 3.986	1.296	- 0.229

TABLE VIII. NORMALIZED STRESS AND DISPLACEMENT COEFFICIENTS FOR A RECTANGULAR PLATE, (β = 1.6 , $\pi h/b$ = 2.545 x 10-2); INELASTIC RESULTS BASED ON ALUMINUM 2024-T3 PLATE MATERIAL

$\frac{e}{(e_{cr})_{cl}}$	$\frac{\sigma_{sv}}{(\sigma_{cr})_{cl}}$	$\frac{A_1}{(e_{cr})_{cl}}$	$\frac{A_2}{(e_{cr})_{cl}}$	$\frac{A_3}{(e_{cr})_{cl}}$	$\frac{B_2}{(e_{cr})_{cl}}$	$\frac{B_3}{(e_{cr})_{cl}}$	$\frac{C}{(e_{cr})_{cl}}$	$\frac{\bar{w}_{11}}{(e_{cr})_{cl}}$	$\frac{\bar{w}_{31}}{(e_{cr})_{cl}} \times 10^2$	$\frac{\bar{w}_{13}}{(e_{cr})_{cl}} \times 10^2$	$\frac{A_{11}}{(e_{cr})_{cl}}$	$\frac{B_{11}}{(e_{cr})_{cl}}$	$\frac{C_{11}}{(e_{cr})_{cl}}$	α
β = 1.6 Elastic Solution B														
2.0	- 1.506	- 1.061	- 1.817	0.800	- 0.952	0.512	- 0.508	0.342	0.163	- 0.908	- 2.191	- 1.640	0.574	- 0.076
3.0	- 1.831	- 0.871	- 3.623	1.242	- 2.345	1.289	- 1.177	0.526	0.582	- 2.875	- 3.354	- 2.546	0.882	- 0.157
4.0	- 2.135	- 0.725	- 4.922	1.151	- 3.906	2.187	- 1.900	0.663	1.164	- 5.069	- 4.207	- 3.247	1.112	- 0.221
5.0	- 2.428	- 0.612	- 5.849	0.668	- 5.621	3.195	- 2.710	0.776	1.881	- 7.267	- 4.903	- 3.847	1.303	- 0.272
6.0	- 2.715	- 0.526	- 6.504	- 0.132	- 7.472	4.303	- 3.616	0.874	2.708	- 9.394	- 5.499	- 4.379	1.469	- 0.313
7.0	- 2.997	- 0.460	- 6.957	- 1.091	- 9.438	5.495	- 4.616	0.962	3.619	-11.430	- 6.024	- 4.861	1.616	- 0.348
8.0	- 3.277	- 0.410	- 7.257	- 2.243	-11.150	6.760	- 5.702	1.041	4.591	-13.370	- 6.493	- 5.303	1.749	- 0.378
β = 1.6 Inelastic Solution														
3.0	- 1.802	- 0.869	- 3.515	1.075	- 2.315	1.268	- 1.057	0.526	0.582	- 3.001	- 3.305	- 2.540	0.880	- 0.154
4.0	- 2.012	- 0.745	- 4.510	1.180	- 3.610	1.910	- 1.400	0.664	1.090	- 5.100	- 3.940	- 3.220	1.080	- 0.204
5.0	- 2.098	- 0.629	- 5.190	1.305	- 4.830	2.580	- 1.791	0.771	1.750	- 8.100	- 4.330	- 3.710	1.210	- 0.234
6.0	- 2.136	- 0.521	- 5.730	1.476	- 5.898	3.142	- 2.093	0.879	2.401	-10.819	- 4.665	- 4.091	1.313	- 0.252
7.0	- 2.163	- 0.440	- 6.140	1.620	- 6.780	3.500	- 2.220	0.960	3.100	-13.200	- 4.800	- 4.310	1.390	- 0.262
8.0	- 2.191	- 0.384	- 6.528	1.847	- 7.350	3.638	- 2.255	1.051	3.696	-15.352	- 4.959	- 4.599	1.448	- 0.270

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APPENDIX I

EULER EQUATIONS AND BOUNDARY CONDITIONS DERIVED FROM THE VANISHING OF THE FIRST VARIATION OF THE REISSNER FUNCTIONAL FOR A TWO-ELEMENT PLATE WITH PRESCRIBED END SHORTENING

The Reissner functional for present considerations is defined as

$$U'' = \int_V \left| \sigma_x \epsilon_x + \sigma_y \epsilon_y + \tau_{xy} \gamma_{xy} - F' \right| dV \tag{26}$$

The strain-displacement relations for small-strain, moderate rotations plate theory, when modified for the two-element plate of Figure 1, become

$$\left.\begin{aligned}
\epsilon_{x_{t,b}} &= \epsilon_x' \pm \epsilon_x'' = u_{,x} + \frac{1}{2} w_{,x}^2 \pm \frac{h}{2} w_{,xx} \\
\epsilon_{y_{t,b}} &= \epsilon_y' \pm \epsilon_y'' = v_{,y} + \frac{1}{2} w_{,y}^2 \pm \frac{h}{2} w_{,yy} \\
\gamma_{xy_{t,b}} &= \gamma_{xy}' \pm \gamma_{xy}'' = u_{,y} + v_{,x} + w_{,x} w_{,y} \pm h w_{,xy}
\end{aligned}\right\} \tag{27}$$

where the + and - signs correspond to the top and bottom elements of the plate, respectively.

The stresses may be written

$$\left.\begin{aligned}
\sigma_{x_{t,b}} &= \sigma_x' \pm \sigma_x'' \\
\sigma_{y_{t,b}} &= \sigma_y' \pm \sigma_y'' \\
\tau_{xy_{t,b}} &= \tau_{xy}' \pm \tau_{xy}''
\end{aligned}\right\} \tag{28}$$

where the primed and double-primed quantities denote in-plane and bending contributions, respectively.

Substitution of Equations (27) and (28) into Equation (26) yields

$$U'' = t_f \int_{-L/2}^{L/2} \int_{-b/2}^{b/2} \left\{ \left(\sigma_{x_t} + \sigma_{x_b}\right)\left(u_{,x} + \frac{1}{2} w_{,x}^2\right) + \left(\sigma_{x_b} - \sigma_{x_b}\right) \frac{h}{2} w_{,xx} \right.$$

$$+ \left(\sigma_{y_t} + \sigma_{y_b}\right)\left(v_{,y} + \frac{1}{2} w_{,y}^2\right) + \left(\sigma_{y_t} - \sigma_{y_b}\right) \frac{h}{2} w_{,yy}$$

$$+ \left(\tau_{xy_t} + \tau_{xy_b}\right)\left(u_{,y} + v_{,x} + w_{,x} w_{,y}\right)$$

$$\left. + \left(\tau_{xy_t} - \tau_{xy_b}\right) h w_{,xy} - \left(F_t' + F_b'\right) \right\} dxdy \tag{29}$$

The average stresses and resulting bending moments are

$$\left.\begin{aligned}
\sigma_x' &= \frac{\left(\sigma_{x_t} + \sigma_{x_b}\right)}{2} \\
\sigma_y' &= \frac{\left(\sigma_{y_t} + \sigma_{y_b}\right)}{2} \\
\tau_{xy}' &= \frac{\left(\tau_{xy_t} + \tau_{xy_b}\right)}{2} \\
M_x &= - t_f \frac{h}{2}\left(\sigma_{x_t} - \sigma_{x_b}\right) = - t_f h \sigma_x'' \\
M_y &= - t_f \frac{h}{2}\left(\sigma_{y_t} - \sigma_{y_b}\right) = - t_f h \sigma_y''
\end{aligned}\right\} \tag{30}$$

(Continued)

$$M_{xy} = t_f \frac{h}{2}\left(\tau_{xy_t} - \tau_{xy_b}\right) = t_f h \tau''_{xy} \Bigg]$$

Substitution of Equation (30) into Equation (20) and with $(F'_t + F'_b)$ replaced by $2F'$ gives

$$U'' = t_f \int_{-L/2}^{L/2} \int_{-b/2}^{b/2} \Bigg\{ 2\sigma'_x \left(u_{,x} + \frac{1}{2} w^2_{,x}\right) + 2\sigma'_y \left(v_{,y} + \frac{1}{2} w^2_{,y}\right)$$

$$+ 2\tau'_{xy}\left(u_{,y} + v_{,x} + w_{,x} w_{,y}\right) - \frac{M_x}{t_f} w_{,xx}$$

$$- \frac{M_y}{t_f} w_{,yy} + 2 \frac{M_{xy}}{t_f} w_{,xy} - 2F' \Bigg\} dxdy \tag{31}$$

In the elastic case,

$$2F' = \frac{1}{2E}\left[\sigma^2_{x_t} + \sigma^2_{y_t} - \sigma_{x_t}\sigma_{y_t} + 3\tau^2_{xy_t} + \sigma^2_{x_b} + \sigma^2_{y_b} - \sigma_{x_b}\sigma_{y_b} + 3\tau^2_{xy_b}\right]$$

or

$$F' = \frac{1}{2E}\left[\sigma'^2_x + \sigma'^2_y - \sigma'_x\sigma'_y + 3\tau'^2_{xy} + \frac{1}{(t_f h)^2}\left(M^2_x + M^2_y - M_x M_y + 3M^2_{xy}\right)\right]$$

whereas in the inelastic case, $F' = \int^{\sigma_{eff}} \epsilon_{eff} d\sigma_{eff}$. The vanishing of the first variation with respect to σ'_x , σ'_y , τ'_{xy} , M_x , M_y , M_{xy} , u , v , and w is then written as

$$\delta U'' = t_f \int_{-L/2}^{L/2} \int_{-b/2}^{b/2} \Bigg\{ 2\delta\sigma'_x \left(u_{,x} + \frac{1}{2} w^2_{,x}\right) + 2\sigma'_x \left(\delta u_{,x} + w_{,x}\delta w_{,x}\right)$$

$$+ 2\delta\sigma'_y \left(v_{,y} + \frac{1}{2} w^2_{,y}\right) + 2\sigma'_y \left(\delta v_{,y} + w_{,y}\, \delta w_{,y}\right)$$

(Continued)

$$+ \; 2\delta\tau'_{xy}\,(u_{,y} + v_{,x} + w_{,x}w_{,y}) + 2\tau'_{xy}\,(\delta u_{,y} + \delta v_{,x}$$

$$+ \; \delta w_{,x}w_{,y} + w_{,x}\delta w_{,y}) - \frac{\delta M_x}{t_f}\,w_{,xx} - \frac{M_x}{t_f}\,\delta w_{,xx}$$

$$- \; \frac{\delta M_y}{t_f}\,w_{,yy} - \frac{M_y}{t_f}\,\delta w_{,yy} + 2\,\frac{\delta M_{xy}}{t_f}\,w_{,xy} + 2\,\frac{M_{xy}}{t_f}\,\delta w_{,xy}$$

$$- \; 2\left[\frac{\partial F'}{\partial \sigma'_x}\,\delta\sigma'_x + \frac{\partial F'}{\partial \sigma'_y}\,\delta\sigma'_y + \frac{\partial F'}{\partial \tau'_{xy}}\,\delta\tau'_{xy} + \frac{\partial F'}{\partial M_x}\,\delta M_x + \frac{\partial F'}{\partial M_y}\,\delta M_y\right.$$

$$\left. + \; \frac{\partial F'}{\partial M_{xy}}\,\delta M_{xy}\right]\Bigg\}\,dxdy \;=\; 0 \qquad (32)$$

With the grouping of like terms and after integration by parts, Equation (32) becomes

$$\delta U'' \;=\; 2t_f \int_{-L/2}^{L/2}\int_{-b/2}^{b/2} \Bigg\{\left(u_{,x} + \frac{1}{2}\,w^2_{,x} - \frac{\partial F'}{\partial \sigma'_x}\right)\delta\sigma'_x$$

$$+ \; \left(v_{,y} + \frac{1}{2}\,w^2_{,y} - \frac{\partial F'}{\partial \sigma'_y}\right)\delta\sigma'_y + \left(u_{,y} + v_{,x} + w_{,x}w_{,y} - \frac{\partial F'}{\partial \tau'_{xy}}\right)\delta\tau'_{xy}$$

$$- \; \left(\frac{w_{,xx}}{2t_f} + \frac{\partial F'}{\partial M_x}\right)\delta M_x - \left(\frac{w_{,yy}}{2t_f} + \frac{\partial F'}{\partial M_y}\right)\delta M_y + \left(\frac{w_{,xy}}{t_f} - \frac{\partial F'}{\partial M_{xy}}\right)\delta M_{xy}$$

$$- \; (\sigma'_{x,x} + \tau'_{xy,y})\;\delta u - (\sigma'_{y,y} + \tau'_{xy,x})\;\delta v$$

$$- \; \Big[(\sigma'_x w_{,x})_{,x} + (\sigma'_y w_{,y})_{,y} + (\tau'_{xy} w_{,y})_{,x} + (\tau'_{xy} w_{,x})_{,y}$$

$$+ \quad \frac{1}{2t_f} \left(M_{x,xx} + M_{y,yy} - 2M_{xy,xy}\right)\Big]\delta w \Bigg\} dxdy$$

$$+ 2t_f \int_{-b/2}^{b/2} \left\{ \sigma_x' \delta u \Big| \begin{matrix} x = + L/2 \\ x = - L/2 \end{matrix} \right\} dy + 2t_f \int_{-L/2}^{L/2} \left\{ \tau_{xy}' \delta u \Big| \begin{matrix} y = + b/2 \\ y = - b/2 \end{matrix} \right\} dx$$

$$+ 2t_f \int_{-L/2}^{L/2} \left\{ \sigma_y' \delta v \Big| \begin{matrix} y = + b/2 \\ y = - b/2 \end{matrix} \right\} dx + 2t_f \int_{-b/2}^{b/2} \left\{ \tau_{xy}' \delta v \Big| \begin{matrix} x = L/2 \\ x = - L/2 \end{matrix} \right\} dy$$

$$+ \int_{-b/2}^{b/2} \left\{ [2t_f \sigma_x' w_{,x} + 2t_f \tau_{xy}' w_{,y} + M_{x,x} - M_{xy,y}] \delta w \Big| \begin{matrix} x = + L/2 \\ x = - L/2 \end{matrix} \right\} dy$$

$$+ \int_{-L/2}^{L/2} \left\{ [2t_f \sigma_y' w_{,y} + 2t_f \tau_{xy}' w_{,x} + M_{y,y} - M_{xy,x}] \delta w \Big| \begin{matrix} y = + b/2 \\ y = - b/2 \end{matrix} \right\} dx$$

$$- \int_{-b/2}^{b/2} \left\{ M_x \delta w_{,x} \Big| \begin{matrix} x = + L/2 \\ x = - L/2 \end{matrix} \right\} dy - \int_{-L/2}^{L/2} \left\{ M_y \delta w_{,y} \Big| \begin{matrix} y = + b/2 \\ y = - b/2 \end{matrix} \right\} dx$$

$$+ \int_{-L/2}^{L/2} \left\{ M_{xy} \delta w_{,x} \Big| \begin{matrix} y = + b/2 \\ y = - b/2 \end{matrix} \right\} dx + \int_{-b/2}^{b/2} \left\{ M_{xy} \delta w_{,y} \Big| \begin{matrix} x = + L/2 \\ x = - L/2 \end{matrix} \right\} dy$$

$$= \quad 0 \tag{33}$$

For Equation (33) to vanish for arbitrary states of stress and displacement consistent with the prescribed displacement boundary conditions, then each of the terms must vanish identically. The Euler equations and boundary conditions that result are obtained as follows:

Stress-displacement relations:

$$\left.\begin{aligned}
\frac{\partial F'}{\partial \sigma_x'} &= \epsilon_x' = u_{,x} + \frac{1}{2} w_{,x}^2 \\
\frac{\partial F'}{\partial \sigma_y'} &= \epsilon_y' = v_{,y} + \frac{1}{2} w_{,y}^2 \\
\frac{\partial F'}{\partial \tau_{xy}'} &= \gamma_{xy}' = u_{,y} + v_{,x} + w_{,x} w_{,y}
\end{aligned}\right\} \qquad (34a)$$

Moment-curvature relations:

$$\left.\begin{aligned}
\frac{\partial F'}{\partial M_x} &= - \frac{1}{2t_f} w_{,xx} \\
\frac{\partial F'}{\partial M_y} &= - \frac{1}{2t_f} w_{,yy} \\
\frac{\partial F'}{\partial M_{xy}} &= \frac{1}{t_f} w_{,xy}
\end{aligned}\right\} \qquad (34b)$$

In-plane equilibrium equations:

$$\sigma_{x,x}' + \tau_{xy,y}' = 0 \qquad (35a)$$

$$\sigma_{y,y}' + \tau_{xy,x}' = 0 \qquad (35b)$$

Out-of-plane equilibrium equation:

$$2t_f\,[\sigma_x' w_{,xx} + 2\tau_{xy}' w_{,xy} + \sigma_y' w_{,yy}] + M_{x,xx} - 2M_{xy,xy} + M_{y,yy} = 0 \qquad (36)$$

Stress (natural) boundary conditions:

$\tau_{xy}'(x, \pm b/2) = 0$	corresponding to arbitrary u-displacement along edges $y = \pm b/2$
$\tau_{xy}'(\pm L/2, y) = 0$	corresponding to arbitrary v-displacement along edges $x = \pm L/2$
$M_x(\pm L/2, y) = 0$	corresponding to arbitrary rotations $w_{,x}$ along edges $x = \pm L/2$
$M_y(x, \pm b/2) = 0$	corresponding to arbitrary rotations $w_{,y}$ along edges $y = \pm b/2$
$\int_{-L/2}^{L/2} \sigma_y'(x, \pm b/2)\,dx = 0$	corresponding to constant v-displacement along edges $y = \pm b/2$

(37)

The Euler equations and boundary conditions established from the variational Equation (33) are the same as those obtained in Reference 1 from potential energy considerations, except for the addition of the stress-displacement and moment-curvature relations given in Equations (34). Naturally, when the constitutive law for the secant-modulus deformation theory of plasticity (see Reference 1) .

$$\left.\begin{aligned} \sigma_x &= \frac{4}{3} E_s \left[\epsilon_x + \frac{1}{2}\epsilon_y\right] \\ \sigma_y &= \frac{4}{3} E_s \left[\epsilon_y + \frac{1}{2}\epsilon_x\right] \\ \tau_{xy} &= \frac{E_s}{3}\gamma_{xy} \end{aligned}\right\} \qquad (38)$$

is introduced into the Reissner functional [Equation (26)] prior to carrying out the variation, the functional can be reduced to the potential energy function of Reference 1. The results obtained from the variation are then identical to those presented in Reference 1.

APPENDIX II

MODIFIED REISSNER FUNCTIONAL FOR ELASTIC PROBLEM

The modified Reissner functional given by Equation (18), after substitution for the assumed stresses [Equation (22)] and displacements [Equation (23)] and subsequent integration over the plate area, becomes

$$\frac{U''}{EV} = -e\left[A_1 + \frac{A_2}{3} + \frac{A_3}{5}\right] + k_1 \left\{ A_1 \frac{\left(\bar{w}_{11}^2 + 9\bar{w}_{31}^2 + \bar{w}_{13}^2\right)}{2} \right.$$

$$+ A_2 \left[k_2 \frac{\left(\bar{w}_{11}^2 + 9\bar{w}_{31}^2\right)}{2} - \frac{3}{2\pi^2} \bar{w}_{11}\bar{w}_{13} + k_{34} \frac{\bar{w}_{13}^2}{2} \right]$$

$$+ A_3 \left[k_3 \frac{\left(\bar{w}_{11}^2 + 9\bar{w}_{31}^2\right)}{2} + k_{35}\bar{w}_{11}\bar{w}_{13} + k_{36} \frac{\bar{w}_{13}^2}{2} \right]$$

$$+ C \left[k_4 \left(k_5 \frac{\bar{w}_{11}^2}{2} + k_6\bar{w}_{31}^2 - k_7\bar{w}_{11}\bar{w}_{31} \right) + \bar{w}_{11}\bar{w}_{13}k_{37} \right.$$

$$\left.\left. + \frac{\bar{w}_{13}^2}{2} k_{38} - \bar{w}_{11}\bar{w}_{31}k_{45} \right] \right\}$$

$$+ k_8 \left\{ -\frac{B_2}{\pi^2} \left[\bar{w}_{11}^2 + 9\bar{w}_{13}^2 + \frac{\bar{w}_{31}^2}{9} + \frac{3}{2} \bar{w}_{11}\bar{w}_{31} \right] \right.$$

$$+ B_3 \left[k_9 \frac{\left(\bar{w}_{11}^2 + 9\bar{w}_{13}^2 + 2\bar{w}_{11}\bar{w}_{31}\right)}{2} + k_{10}\bar{w}_{11}\bar{w}_{31} + k_{11} \frac{\bar{w}_{31}^2}{2} \right]$$

$$+ C \left[\frac{k_{12}}{\pi^2} \left(\bar{w}_{11}^2 + \frac{\bar{w}_{31}^2}{9} + \frac{3}{2} \bar{w}_{11}\bar{w}_{31} \right) + k_{39}\bar{w}_{13}^2 \right.$$

(Continued)

$$+ k_{40}\left(6\bar{w}_{11}\bar{w}_{13} + \frac{9}{2}\bar{w}_{13}\bar{w}_{31}\right)\Bigg]\Bigg\}$$

$$- Ck_{13}\left\{k_{14}\left(\bar{w}_{11}^2 + \frac{\bar{w}_{31}^2}{9} + \frac{3}{2}\bar{w}_{11}\bar{w}_{31}\right) + k_{41}\bar{w}_{13}^2 + k_{42}\bar{w}_{11}\bar{w}_{13}\right.$$

$$\left. + k_{43}\bar{w}_{13}\bar{w}_{31}\right\}$$

$$- \frac{1}{2}\left[A_1^2 + \frac{A_2^2}{5} + \frac{A_3^2}{9} + C^2 (k_{15} + k_{20} + k_{24} - k_{27}) + \frac{2}{3}A_1A_2\right.$$

$$+ \frac{2}{5}A_1A_3 + \frac{2}{7}A_2A_3 - C (k_{16}A_2 + k_{17}A_3 + k_{22}B_2 + k_{23}B_3)$$

$$\left. + k_{18}B_2^2 + k_{19}B_3^2 + k_{21}B_2B_3\right]$$

$$+ k_{25}\bar{w}_{11}^2 + k_{26}\bar{w}_{31}^2 + k_{44}\bar{w}_{13}^2 \tag{39}$$

where

$$\left.\begin{aligned} \bar{w}_{11} &= \left(\frac{\pi h}{b}\right) w_{11} \\ \bar{w}_{31} &= \left(\frac{\pi h}{b}\right) w_{31} \\ \bar{w}_{13} &= \left(\frac{\pi h}{b}\right) w_{13} \end{aligned}\right\} \tag{40}$$

The coefficients k_1 through k_{45} are numerical constants resulting from the integration process. The constants, some of which depend upon either or both of the parameters b/λ and $\pi h/b$, are

k_1	=	$2.46739\,[\beta^2/(\pi h/b)^2]$	k_{17}	=	0.01975
k_2	=	0.13069	k_{18}	=	0.08889
k_3	=	0.04110	k_{19}	=	0.07111
k_4	=	0.06219	k_{20}	=	$0.00054\,\beta^4$
k_5	=	0.35653	k_{21}	=	0.15238
k_6	=	1.05684	k_{22}	=	$0.00931\,\beta^2$
k_7	=	0.39268	k_{23}	=	$0.00798\,\beta^2$
k_8	=	$2.46739\,[1/(\pi h/b)^2]$	k_{24}	=	$0.00117\,\beta^2$
k_9	=	-0.15890	k_{25}	=	$0.41123\,\beta^4 + 2\beta^2 + 1$
k_{10}	=	0.08592	k_{26}	=	$0.41123\,(81\beta^4 + 18\beta^2 + 1)$
k_{11}	=	-0.04199	k_{27}	=	$0.00035\,\beta^2$
k_{12}	=	$0.09019\,\beta^2$	k_{34}	=	0.31082
k_{13}	=	$0.31831\,[\beta^2/(\pi h/b)^2]$	k_{35}	=	-0.07298
k_{14}	=	0.05939	k_{36}	=	0.15801
k_{15}	=	0.00276	k_{37}	=	-0.0108
k_{16}	=	0.17780	k_{38}	=	0.01692

(Continued)

$$k_{39} = 0.05069\ \beta^2 \qquad k_{43} = 0.19490$$

$$k_{40} = -0.00526\ \beta^2 \qquad k_{44} = 0.41123\ (\beta^4 + 18\ \beta^2 + 81)$$

$$k_{41} = 0.04533 \qquad k_{45} = -0.01192$$

$$k_{42} = -0.05797 \tag{41}$$

The requirement of the vanishing of the first variation of the functional, given by Equation (39) with respect to the arbitrary parameters, leads to a set of nonlinear algebraic equations, the solution of which establishes the values of the stress coefficients A_1 , A_2 , A_3 , B_2 , B_3 , and C and the displacement coefficients $\bar{w}_{11}$, $\bar{w}_{31}$, and $\bar{w}_{13}$. The results obtained with $\bar{w}_{13} = 0$ are designated Solutions A1, whereas the results obtained with $\bar{w}_{13}$ retained are designated Solutions A2.

APPENDIX III

UNMODIFIED REISSNER FUNCTIONAL FOR ELASTIC PROBLEM

The unmodified Reissner functional for elastic problems given by Equation (14), after substitution for the assumed stresses [Equations (22) and (24)] and displacements [Equation (23)] and subsequent integration over the plate area, becomes

$$\frac{U''}{EV} = -e\left[A_1 + \frac{A_2}{3} + \frac{A_3}{5}\right] + k_1\left\{A_1\left[\frac{\bar{w}_{11}^2 + 9\bar{w}_{31}^2 + \bar{w}_{13}^2}{2}\right]\right.$$

$$+ A_2\left[k_2\frac{\left(\bar{w}_{11}^2 + 9\bar{w}_{31}^2\right)}{2} - \frac{3}{2\pi^2}\bar{w}_{11}\bar{w}_{13} + \frac{\bar{w}_{13}^2}{2}k_{34}\right]$$

$$+ A_3\left[k_3\frac{\left(\bar{w}_{11}^2 + 9\bar{w}_{31}^2\right)}{2} + k_{35}\bar{w}_{11}\bar{w}_{13} + \frac{\bar{w}_{13}^2}{2}k_{36}\right]$$

$$+ C\left[k_4\left(\frac{\bar{w}_{11}^2}{2} + k_6\bar{w}_{31}^2 - k_7\bar{w}_{11}\bar{w}_{31}\right) + k_{37}\bar{w}_{11}\bar{w}_{13}\right.$$

$$\left.\left. + k_{38}\frac{\bar{w}_{13}^2}{2} - k_{45}\bar{w}_{13}\bar{w}_{31}\right]\right\}$$

$$+ k_8\left\{-\frac{B_2}{\pi^2}\left[\bar{w}_{11}^2 + 9\bar{w}_{13}^2 + \frac{\bar{w}_{31}^2}{9} + \frac{3}{2}\bar{w}_{11}\bar{w}_{31}\right]\right.$$

$$+ B_3\left[k_9\frac{\left(\bar{w}_{11}^2 + 9\bar{w}_{13}^2 + 2\bar{w}_{11}\bar{w}_{31}\right)}{2} + k_{10}\bar{w}_{11}\bar{w} + k_{11}\frac{\bar{w}_{31}^2}{2}\right]$$

(Continued)

$$
\begin{aligned}
&+ C\left[\frac{k_{12}}{\pi^2}\left(\bar{w}_{11}^2 + \frac{\bar{w}_{31}^2}{9} + \frac{3}{2}\,\bar{w}_{11}\bar{w}_{31}\right) + k_{39}\bar{w}_{13}^2 + k_{40}\left(6\,\bar{w}_{11}\bar{w}_{13}\right.\right. \\
&\left.\left.\left. + \frac{9}{2}\,\bar{w}_{13}\bar{w}_{31}\right)\right]\right\} \\
&- Ck_{13}\left[k_{14}\left(\bar{w}_{11}^2 + \frac{\bar{w}_{31}^2}{9} + \frac{3}{2}\,\bar{w}_{11}\bar{w}_{31}\right) + k_{41}\bar{w}_{13}^2 + k_{42}\bar{w}_{11}\bar{w}_{13}\right. \\
&\left. + k_{43}\bar{w}_{13}\bar{w}_{31}\right] - \frac{\pi}{8}\,\beta^2 A_{11}\,(\bar{w}_{11} + \alpha\,\bar{w}_{13}) - \frac{\pi}{8}\,B_{11}\,(\bar{w}_{11} + 9\,\alpha\,\bar{w}_{13}) \\
&+ \frac{\pi}{4}\,C_{11}\,\beta\,(\bar{w}_{11} + 3\,\alpha\,\bar{w}_{13}) \\
&- \frac{1}{2}\left[A_1^2 + \frac{A_2^2}{5} + \frac{A_3^2}{9} + C^2\,(k_{15} + k_{20} + k_{24} - k_{27})\right. \\
&+ \left(\frac{1 + \alpha^2}{4}\right)(A_{11}^2 + B_{11}^2 - A_{11}B_{11} + 3C_{11}^2) \\
&+ B_2^2k_{18} + B_3^2k_{19} + \frac{2}{3}\,A_1A_2 + \frac{2}{5}\,A_1A_3 + \frac{2}{7}\,A_2A_3 \\
&\left. - C\,(A_2k_{16} + A_3k_{17} + B_2k_{22} + B_3k_{23}) + B_2B_3k_{21}\right] \qquad (42)
\end{aligned}
$$

where $\bar{w}_{11}$, $\bar{w}_{31}$, and $\bar{w}_{13}$ are given by Equations (40) and the k coefficients, by Equation (41).

The results obtained from the simultaneous vanishing of the first variation of the functional given in Equation (42) with respect to the stress coefficients A_1 , A_2 , A_3 , B_2 , B_3 , C , A_{11} , B_{11} , C_{11} , and α and the displacement coefficients $\bar{w}_{11}$, $\bar{w}_{31}$, and $\bar{w}_{13}$ are designated Solutions B.

APPENDIX IV
NEWTON-RAPHSON ITERATIVE TECHNIQUE

The requirement that the Reissner functionals given in Equations (39), (42), and (25) have a stationary value results in sets of simultaneous nonlinear algebraic equations, the solution of which establishes the values of the stress and displacement coefficients. The basic Newton-Raphson technique is employed to effect the solution. The method is illustrated by the following example. Consider a function $F(x_1, x_2, \ldots x_m)$ for which the values of x_i are sought that render F stationary. This requires that

$$\frac{\partial F(x_i)}{\partial x_i} \equiv \psi_i(x_i) = 0 \qquad i = 1,2,\ldots m \text{ (number of independent variables)}$$

If approximate values of x_i are known, the above relations may not be satisfied. Therefore, by applying a suitable correction Δx_i such that $\psi_i(x_i + \Delta x_i) = 0$, the desired values of x_i are obtained. To determine the correction Δx_i, the functions $\psi_i(x_i + \Delta x_i)$ are expanded into a Taylor series about x_i with only the linear terms retained. This procedure results in the following system of linear algebraic equations

$$\psi_i(x_i + \Delta x_i) = \psi_i(x_i) + \sum_{j=1}^{m} \frac{\partial \psi_i}{\partial x_j} \Delta x_j = 0 \tag{43}$$

where $i = 1,2,3\ldots m$. Written in matrix form, the equations are

$$\left[\frac{\partial \psi_i}{\partial x_j} = \frac{\partial^2 F}{\partial x_i \partial x_j} \right] \left\{ \Delta x_i \right\} = -\left\{ \frac{\partial F}{\partial x_i} \right\} \tag{44}$$

where $i = 1,2\ldots m$ and $j = 1,2\ldots m$. Thus, by adding the correction Δx_i to the starting value x_i and by repeating the process until $\partial F/\partial x_i \rightarrow 0$, the solution is obtained.

For the inelastic problem formulated in Equation (25), the partial derivatives indicated previously are obtained numerically, whereas, in the elastic solutions, the derivatives are evaluated explicitly. The partial derivative approximations utilized are

$$\frac{\partial F}{\partial x_i} = \frac{F(x_i + \delta x_i) - F(x_i - \delta x_i)}{2\delta x_i}$$

$$\frac{\partial^2 F}{\partial x_i^2} = \frac{F(x_i + \delta x_i) - 2F(x_i) + F(x_i - \delta x_i)}{(\delta x_i)^2}$$

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{F(x_i + \delta x_i\ ,\ x_j + \delta x_j) - F(x_i + \delta x_i) - F(x_j + \delta x_j) + F(x_i, x_j)}{\delta x_i \delta x_j} \tag{45}$$

where $i = 1,2 \ldots m$ and $j = 1,2 \ldots m$.

The step sizes δx_i used in evaluating these derivatives are taken as $x_i \times 10^{-3}$. The area integration indicated in Equation (25) is performed numerically utilizing a basic trapezoidal technique carried out on a 5×5 plate grid.

Unclassified

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)

Stanford University
Department of Aeronautics and Astronautics
Stanford, California

2a. REPORT SECURITY CLASSIFICATION

Unclassified

2b. GROUP

N/A

3. REPORT TITLE

ELASTIC AND MAXIMUM STRENGTH ANALYSES OF POSTBUCKLED RECTANGULAR PLATES BASED UPON MODIFIED VERSIONS OF REISSNER'S VARIATIONAL PRINCIPLE

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Final Technical Report

5. AUTHOR(S) (First name, middle initial, last name)

Mayers, Jean
Nelson, Edward

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
July 1970	74	11

8a. CONTRACT OR GRANT NO.

DA 44-177-AMC-115(T)

b. PROJECT NO

Task 1F162204A17002

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S)

USAAVLABS Technical Report 69-64

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

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U.S. Army Aviation Materiel Laboratories
Fort Eustis, Virginia

13. ABSTRACT

The postbuckling and maximum strength analyses of uniformly shortened, simply supported rectangular plates with straight, unloaded edges are performed by using Reissner's variational principle in conjunction with a deformation theory of plasticity. The results are compared to (1) experimental data for a rectangular plate in which the test conditions basically reflect the boundary conditions specified in the present analyses, and (2) a potential energy solution that correlates well with experiment, but in which the effect of waveform change on the average stress carried by the plate is accounted for only in a gross manner and in which the effect of small, local unloading is neglected. Good agreement with experiment establishes confidence in the new approach and indicates that the simplified technique utilized in the potential energy solution to compensate for waveform changes may be employed for engineering purposes. Both analytical approaches predict a slightly conservative plate maximum strength relative to the experimental result. This discrepancy is attributed to slight departures of the test conditions from the ideal boundary and loading conditions assumed in the analyses.

An application of Reissner's principle and a modified version of the principle is undertaken initially to obtain the elastic postbuckling behavior of uniaxially compressed square and rectangular plates. Excellent agreement of elastic-solution results with essentially "exact" solutions of other authors for the same boundary conditions establishes the effectiveness of the Reissner and modified Reissner principles and justifies the application of the Reissner-principle approach to the maximum strength problem.

DD FORM 1 NOV 65 1473

Unclassified

Security Classification

Unclassified

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Plates						
Stability						
Maximum Strength						
Failure Analysis						
Plasticity						

Unclassified

Security Classification